Basic superranks for varieties of algebras

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Abstract

We introduce the notion of basic superrank for varieties of algebras generalizing the notion of basic rank. First we consider a number of varieties of nearly associative algebras over a field of characteristic 0 that have infinite basic ranks and calculate their basic superranks which turns out to be finite. Namely we prove that the variety of alternative metabelian (solvable of index 2) algebras possesses the two basic superranks $(1,1)$ and $(0,3)$; the varieties of Jordan and Malcev metabelian algebras have the unique basic superranks $(0,2)$ and $(1,1)$, respectively. Furthermore, for arbitrary pair $(r,s)\neq(0,0)$ of nonnegative integers we provide a variety that has the unique basic superrank (r,s). Finally, we construct some examples of nearly associative varieties that do not possess finite basic superranks.

Key words: alternative algebra, Jordan algebra, Malcev algebra, metabelian algebra, Grassmann algebra, superalgebra, variety of algebras, basic rank of variety, basic superrank of variety, basic spectrum of variety.

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Dedicated to Efim Isaakovich Zelmanov, on the occasion of his 60th birthday

Introduction

Throughout the paper, all algebras are considered over a field of characteristic 0. Let $\mathcal{V}$ be a variety of algebras and $\mathcal{V}_r$ be a subvariety of $\mathcal{V}$ generated by the free $\mathcal{V}$-algebra of rank r. Then one can consider the chain

$$\mathcal{V}_1\subseteq\mathcal{V}_2\subseteq\cdots\subseteq\mathcal{V}_r\subseteq\cdots\subseteq\mathcal{V},$$

where $\mathcal{V}=\bigcup_r\mathcal{V}_r$. If this chain stabilizes, then the minimal number r with the property $\mathcal{V}_r=\mathcal{V}$ is called the *basic rank* of the variety $\mathcal{V}$ and is denoted by $r_b(\mathcal{V})$ (see [13]). Otherwise, we say that $\mathcal{V}$ has the *infinite basic rank* $r_b(\mathcal{V})=\aleph_0$.

Let us recall the main results on the basic ranks of the *varieties of associative* (Assoc), *Lie* (Lie), *alternative* (Alt), *Malcev* (Malc), and some other *algebras*. It was first shown

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by A. I. Mal'cev [13] that $r_b\,(\mathrm{Assoc}) = 2$. A. I. Shirshov [28] proved that $r_b\,(\mathrm{Lie}) = 2$ and $r_b\,(\mathrm{SJord}) = 2$, where SJord is the variety generated by all special Jordan algebras. In 1958, A. I. Shirshov posed a problem on finding basic ranks for alternative and some other varieties of nearly associative algebras [3, Problem 1.159]. In 1977, the second author proved that $r_b\,(\mathrm{Alt}) = r_b\,(\mathrm{Malc}) = \aleph_0$ [18, 28]. The similar fact for the variety of algebras of type $(-1,1)$ was established by S. V. Pchelintsev [14]. Note that the basic ranks of the varieties of Jordan and right alternative algebras are still unknown.

A proper subvariety of associative algebras can be of infinite basic rank as well. For instance, so is the variety $\mathrm{Var}\,\mathrm{G}$ generated by the Grassmann algebra G on infinite number of generators, or the variety defined by the identity $[x,y]^n = 0,\ n > 1$.

In 1986, A. R. Kemer [10, 11] solved affirmatively the famous Specht problem [21] using the tool of superalgebras. Recall that a variety $\mathcal{V}$ of algebras is called *Spechtian* or is said *to have the Specht property* if every algebra of $\mathcal{V}$ possesses a finite basis for its identities. The Kemer Theorem states the Specht property of the variety of associative algebras. This result is obtained by certain reduction to the case of graded identities of finite dimensional superalgebras. Namely it is proved that the ideal of identities of arbitrary associative algebra coincides with the ideal of identities of the Grassmann envelope of some finite dimensional superalgebra.

This result suggests a generalization of the notion of basic rank. Namely we shall say that a variety $\mathcal{V}$ has a finite basic superrank if it can be generated by the Grassmann envelope of some finitely generated superalgebra. Then Kemer's result implies that every variety of associative algebras has a finite basic superrank. This means that the notion of basic superrank is a more refined one then that of basic rank, and we can distinguish varieties of infinite basic rank by their superranks.

The notion of basic superrank is the main subject of our paper. First we consider a number of varieties of nearly associative algebras over a field of characteristic 0 that have infinite basic ranks and calculate their basic superranks which turns out to be finite. Namely we prove that the variety of alternative metabelian (solvable of index 2) algebras possesses the two basic superranks $(1,1)$ and $(0,3)$; the varieties of Jordan and Malcev metabelian algebras have the unique basic superranks $(0,2)$ and $(1,1)$, respectively. Furthermore, for arbitrary pair $(r,s) \neq (0,0)$ of nonnegative integers we provide a variety that has the unique basic superrank (r,s). Finally, we construct some examples of nearly associative varieties of algebras that do not possess finite basic superranks.

1 Main definitions and results

Let $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$ be a *superalgebra* ($\mathbb{Z}_2$-graded algebra) with the *even part* $\mathcal{A}_0$ and the *odd part* $\mathcal{A}_1$, i. e. $\mathcal{A}_i\mathcal{A}_j \subseteq \mathcal{A}_{i+j\ (\mathrm{mod} 2)}$ for $i,j \in \{0,1\}$; G be the *Grassmann algebra* on a countable set of anticommuting generators $\{e_1, e_2, \ldots \mid e_ie_j = -e_je_i\}$ with the natural $\mathbb{Z}_2$-grading (G_0 and G_1 are spanned by the words of even and, respectively, odd length on $\{e_i\}$). The *Grassmann envelope* $\mathrm{G}\,(\mathcal{A})$ of a superalgebra $\mathcal{A}$ is the subalgebra $\mathrm{G}_0 \otimes \mathcal{A}_0 + \mathrm{G}_1 \otimes \mathcal{A}_1$ of the tensor product $\mathrm{G} \otimes \mathcal{A}$.

It is well known that $\mathrm{G}\,(\mathcal{A})$ satisfies a multilinear polynomial identity $f = 0$ if and only if $\mathcal{A}$ satisfies the graded identity $\tilde{f} = 0$ called the *superization of* $f = 0$. Here,

$\tilde{f}$ denotes the so-called *superpolynomial corresponding to* f and we say that $\mathcal{A}$ satisfies the *superidentity* $\tilde{f} = 0$. The detailed descriptions of the process of constructing of superpolynomials (the *superizing process*) can be found in [19, 20, 24, 27]. Roughly speaking, one should apply the so-called *Koszul rule* (or *Kaplansky rule*): one should introduce the sign $(-1)^{ij}$ always when a variable of parity i passes through a variable of parity j.

Let $\mathcal{V}$ be a variety of algebras defined by a system S of multilinear identities. Recall that $\mathcal{A}$ is said to be a *$\mathcal{V}$-superalgebra* if its Grassmann envelope lies in $\mathcal{V}$, i. e. if $\mathcal{A}$ satisfies the system $\tilde{S}$ of all superidentities corresponding to the defining identities of $\mathcal{V}$. Thus one can consider the set $\tilde{\mathcal{V}}$ of all $\mathcal{V}$-superalgebras as a *supervariety defined by the system* $\tilde{S}$. It is clear that $\tilde{\mathcal{V}}$ can be generated by the free $\mathcal{V}$-superalgebra on a countable set of even and a countable set of odd generators. Let $\mathcal{V}_{r,s}$ be the *supervariety generated by the free $\mathcal{V}$-superalgebra on r even and s odd generators* for $(r,s) \neq (0,0)$ and $\mathcal{V}_{0,0} = \big\{\{0\}\big\}$. By $\mathcal{L}(\mathcal{V})$ denote the lattice

$$\begin{array}{ccccccccc}
 & & \mathcal{V}_{1,0} & \subseteq & \mathcal{V}_{2,0} & \subseteq \cdots \subseteq & \mathcal{V}_{r,0} & \subseteq \cdots \\
 & & \cap\!\!\!| & & \cap\!\!\!| & & \cap\!\!\!| & \\
\mathcal{V}_{0,1} & \subseteq & \mathcal{V}_{1,1} & \subseteq & \mathcal{V}_{2,1} & \subseteq \cdots \subseteq & \mathcal{V}_{r,1} & \subseteq \cdots \\
\cap\!\!\!| & & \cap\!\!\!| & & \cap\!\!\!| & & \cap\!\!\!| & \\
\vdots & & \vdots & & \vdots & & \vdots & \\
\cap\!\!\!| & & \cap\!\!\!| & & \cap\!\!\!| & & \cap\!\!\!| & \\
\mathcal{V}_{0,s} & \subseteq & \mathcal{V}_{1,s} & \subseteq & \mathcal{V}_{2,s} & \subseteq \cdots \subseteq & \mathcal{V}_{r,s} & \subseteq \cdots \\
\cap\!\!\!| & & \cap\!\!\!| & & \cap\!\!\!| & & \cap\!\!\!| & \\
\multicolumn{8}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}
\end{array}$$

It is clear that $\mathcal{L}(\mathcal{V})$ is partially well-ordered with the relation of inclusion. A pair (r,s) is called the *basic superrank of the variety* $\mathcal{V}$ if $\mathcal{V}_{r,s}$ is a minimal element of $\mathcal{L}(\mathcal{V})$ with the property $\mathcal{V}_{r,s} = \tilde{\mathcal{V}}$. If $\mathcal{V}_{r,s} \neq \tilde{\mathcal{V}}$ for all r, s, we say that the *basic superrank of* $\mathcal{V}$ *is infinite.* The set of all possible finite basic superranks of $\mathcal{V}$ is called the *basic spectrum of* $\mathcal{V}$ and is denoted by $\mathrm{sp_b}(\mathcal{V})$. For instance, one can easily check that $\mathrm{sp_b}(\mathrm{Assoc}) = \mathrm{sp_b}(\mathrm{Lie}) = \{(2,0),(1,1),(0,2)\}$. It is not hard to prove that if $\mathcal{V}$ possesses at least one finite basic superrank (r,s), then $\mathrm{sp_b}(\mathcal{V})$ is a finite set of at most $r+s+1$ elements.

The first examples of varieties of infinite basic superrank were constructed by M. V. Zaitsev [25, 26].

Let us fix some notations. While writing down nonassociative monomials we use the symbol $\cdot$ instead of parentheses to indicate the correct order of multiplication. For instance, we write $xy \cdot z$ instead of $(xy)z$ and $x \cdot yz$ instead of $x(yz)$. By

$$[x,y] = xy - yx \quad \text{and} \quad x \circ y = xy + yx$$

we denote, respectively, the *commutator* and the *Jordan product* of the elements x, y. The notations

$$(x,y,z) = xy \cdot z - x \cdot yz \quad \text{and} \quad \mathrm{J}(x,y,z) = xy \cdot z + yz \cdot x + zx \cdot y$$

are used, respectively, for the *associator* and the *Jacobian* of the elements x, y, z.

Recall that the *varieties* Alt, Jord, and Malc *of alternative, Jordan,* and *Malcev algebras* are defined by the following pairs of identities:

$$\mathrm{Alt}: \quad (x,x,y)=0, \quad (x,y,y)=0; \tag{1}$$

$$\mathrm{Jord}: \quad [x,y]=0, \quad \big(x^2,y,x\big)=0; \tag{2}$$

$$\mathrm{Malc}: \quad x\circ y=0, \quad \mathrm{J}\,(x,y,z)\,x=\mathrm{J}\,(x,y,xz)\,. \tag{3}$$

By $\mathcal{V}^{(2)}$ we denote a *subvariety of all metabelian (solvable of index at most* 2*) algebras of a given variety* $\mathcal{V}$, i. e. $\mathcal{V}^{(2)}$ is distinguished in $\mathcal{V}$ by the identity

$$xy\cdot zt=0. \tag{4}$$

By NAlt we denote a subvariety of all nilalgebras in Alt of index at most 3, i. e. NAlt is distinguished in Alt by the identity

$$x^3=0. \tag{5}$$

Let us formulate the results of the paper. First, in Section 2, we describe the inclusions in the lattices $\mathcal{L}\big(\mathrm{Alt}^{(2)}\big)$ and $\mathcal{L}\big(\mathrm{NAlt}^{(2)}\big)$ and calculate the basic spectrums of these varieties. Namely, we prove the following theorems.

Theorem 1. $\mathrm{sp_b}\big(\mathrm{NAlt}^{(2)}\big)=\big\{(0,1)\big\}$ *and all inclusions* $\mathrm{NAlt}^{(2)}_n\subset\mathrm{NAlt}^{(2)}_{n+1}$ *are strict.*

Theorem 2. *For* $\mathcal{V}=\mathrm{Alt}^{(2)}$ *we have*

$$\begin{array}{ccccccccc}
 & & \mathcal{V}_{1,0} & \subset & \mathcal{V}_{2,0} & \subset\cdots\subset & \mathcal{V}_{r,0} & \subset\cdots \\
 & & \cap & & \cap & & \cap & \\
\mathcal{V}_{0,1} & \subset & \mathcal{V}_{1,1} & = & \mathcal{V}_{2,1} & =\cdots= & \mathcal{V}_{r,1} & =\cdots \\
\parallel & & \parallel & & \parallel & & \parallel & \\
\mathcal{V}_{0,2} & \subset & \mathcal{V}_{1,2} & = & \mathcal{V}_{2,2} & =\cdots= & \mathcal{V}_{r,2} & =\cdots \\
\cap & & \parallel & & \parallel & & \parallel & \\
\mathcal{V}_{0,3} & = & \mathcal{V}_{1,3} & = & \mathcal{V}_{2,3} & =\cdots= & \mathcal{V}_{r,3} & =\cdots \\
\parallel & & \parallel & & \parallel & & \parallel & \\
\mathcal{V}_{0,4} & = & \mathcal{V}_{1,4} & = & \mathcal{V}_{2,4} & =\cdots= & \mathcal{V}_{r,4} & =\cdots \\
\parallel & & \parallel & & \parallel & & \parallel & \\
\multicolumn{8}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots}
\end{array}$$

Corollary 2.1. $\mathrm{sp_b}\big(\mathrm{Alt}^{(2)}\big)=\big\{(1,1),(0,3)\big\}$.

In Sections 3 and 4, we describe the inclusions in the lattices $\mathcal{L}\big(\mathrm{Jord}^{(2)}\big)$, $\mathcal{L}\big(\mathrm{Malc}^{(2)}\big)$ and calculate the basic spectrums $\mathrm{sp_b}\big(\mathrm{Jord}^{(2)}\big)$, $\mathrm{sp_b}\big(\mathrm{Malc}^{(2)}\big)$, respectively.

Theorem 3. *For* $\mathcal{V}=\mathrm{Jord}^{(2)}$ *we have*

$$\begin{array}{ccccccccc}
 & & \mathcal{V}_{1,0} & \subset & \mathcal{V}_{2,0} & \subset\cdots\subset & \mathcal{V}_{r,0} & \subset\cdots \\
 & & \cap & & \cap & & \cap & \\
\mathcal{V}_{0,1} & \subset & \mathcal{V}_{1,1} & \subset & \mathcal{V}_{2,1} & \subset\cdots\subset & \mathcal{V}_{r,1} & \subset\cdots \\
\cap & & \cap & & \cap & & \cap & \\
\mathcal{V}_{0,2} & = & \mathcal{V}_{1,2} & = & \mathcal{V}_{2,2} & =\cdots= & \mathcal{V}_{r,2} & =\cdots \\
\parallel & & \parallel & & \parallel & & \parallel & \\
\mathcal{V}_{0,3} & = & \mathcal{V}_{1,3} & = & \mathcal{V}_{2,3} & =\cdots= & \mathcal{V}_{r,3} & =\cdots \\
\parallel & & \parallel & & \parallel & & \parallel & \\
\multicolumn{8}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots}
\end{array}$$

Corollary 3.1. $\mathrm{sp_b}\big(\mathrm{Jord}^{(2)}\big) = \big\{(0,2)\big\}$.

Theorem 4. *For* $\mathcal{V} = \mathrm{Malc}^{(2)}$ *we have*

$$\begin{array}{ccccccccccc}
 & & \mathcal{V}_{1,0} & \subset & \mathcal{V}_{2,0} & \subset \cdots \subset & \mathcal{V}_{r,0} & \subset \cdots \\
 & & \cap & & \cap & & \cap & \\
\mathcal{V}_{0,1} & \subset & \mathcal{V}_{1,1} & = & \mathcal{V}_{2,1} & = \cdots = & \mathcal{V}_{r,1} & = \cdots \\
\cap & & \shortparallel & & \shortparallel & & \shortparallel & \\
\mathcal{V}_{0,2} & \subset & \mathcal{V}_{1,2} & = & \mathcal{V}_{2,2} & = \cdots = & \mathcal{V}_{r,2} & = \cdots \\
\cap & & \shortparallel & & \shortparallel & & \shortparallel & \\
\vdots & & \vdots & & \vdots & \ddots & \vdots & \\
\cap & & \shortparallel & & \shortparallel & & \shortparallel & \\
\mathcal{V}_{0,s} & \subset & \mathcal{V}_{1,s} & = & \mathcal{V}_{2,s} & = \cdots = & \mathcal{V}_{r,s} & = \cdots \\
\cap & & \shortparallel & & \shortparallel & & \shortparallel & \\
\multicolumn{8}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}
\end{array}$$

Corollary 4.1. $\mathrm{sp_b}\big(\mathrm{Malc}^{(2)}\big) = \big\{(1,1)\big\}$.

Further, let $\mathfrak{M}$ be the *variety of all metabelian algebras.* In Section 5, we describe the inclusions in the lattice $\mathcal{L}(\mathfrak{M})$ and provide an example of variety of unique arbitrary given finite basic superrank.

Theorem 5. *The inclusion* $\mathfrak{M}_{r',s'} \subseteq \mathfrak{M}_{r,s}$ *holds only if* $r' \leqslant r$ *and* $s' \leqslant s$. *The equality* $\mathfrak{M}_{r',s'} = \mathfrak{M}_{r,s}$ *takes place only if* $r' = r$ *and* $s' = s$.

Corollary 5.1. *The basic superrank of* $\mathfrak{M}$ *is infinite.*

Corollary 5.2. *For an arbitrary pare* $(r,s) \neq (0,0)$ *of nonnegative integers, the variety of algebras generated by the Grassmann envelope of the free* $\mathfrak{M}_{r,s}$*-superalgebra has the unique basic superrank* (r,s).

In Section 6, we provide some examples of varieties of nearly associative algebras of infinite basic superrank. For $\varepsilon = \pm 1$, by $\mathcal{V}^{\langle\varepsilon\rangle}$ denote a subvariety of $\mathfrak{M}$ distinguished by the identities

$$(x,y,z) = \varepsilon\,(x,z,y)\,, \tag{6}$$

$$\big\langle \langle x,y\rangle_\varepsilon\,, z\big\rangle_\varepsilon = 0, \tag{7}$$

where $\langle x,y\rangle_\varepsilon = xy - \varepsilon yx$. Thus, $\mathcal{V}^{\langle +1\rangle}$ is the *variety of metabelian right symmetric algebras that are Lie-nilpotent of index at most* 2 and $\mathcal{V}^{\langle -1\rangle}$ is the *variety of metabelian right alternative algebras that are Jordan-nilpotent of index at most* 2.

Theorem 6. *The basic superrank of* $\mathcal{V}^{\langle\varepsilon\rangle}$ *is infinite.*

Finally, in Section 7, we suggest some open problems dealing with the introduced notions of basic superrank and basic spectrum for varieties of algebras.

Common notations

Throughout the paper, we use the following notations. By $[r]$ we denote the *integer part of number* r; R_x and L_x are *operators of right* and *left multiplication*, respectively, *by an element* x; T_x is a common notation for R_x and L_x; $T_x^* = \begin{cases} L_x, & \text{if } T_x = R_x, \\ R_x, & \text{if } T_x = L_x; \end{cases}$ $X = \{x_1, x_2, \ldots\}$ is a *countable set*; $X_n = \{x_1, x_2, \ldots, x_n\}$, $n \in \mathbb{N}$; F is a *field of characteristic* $\operatorname{char} F = 0$; $F_{\mathcal{V}}[Y]$ is a *free algebra of variety* $\mathcal{V}$ *on a set* Y *of free generators over* F; $F_{\mathcal{V}}^{(s)}[Z]$ is a *free* $\mathcal{V}$*-superalgebra on a set* Z *of free even and odd generators over* F; $\mathcal{P}_n(\mathcal{V})$ is a *subspace of* $F_{\mathcal{V}}[X_n]$ *of all multilinear polynomials of degree* $n \geqslant 2$; $(f)^{\mathrm{T}}$ is a T*-ideal of algebra* $F_{\mathcal{V}}[X]$ *generated by the given polynomial* f; S_n is the *symmetric group on the set* $1, 2, \ldots, n$; A_n is the *alternating subgroup of* S_n; C_n is the *subgroup of* S_n *generated by the cycle* $(1\,2\ldots n)$; $|\sigma|$ is a *parity of permutation* $\sigma \in \mathrm{S}_n$, i. e. $|\sigma| = \begin{cases} 0, & \text{if } \sigma \text{ is even}, \\ 1, & \text{if } \sigma \text{ is odd}. \end{cases}$

In order to avoid complicated formulas while writing down the elements of the space $\mathcal{P}_n(\mathcal{V})$ we omit the indices of variables at the operator symbols R, L and assume them to be arranged in the ascending order. For example, the notation $(x_2x_4)\,LR^2$ means the monomial $(x_2x_4)\,L_{x_1}R_{x_3}R_{x_5}$.

2 Alternative algebras

Throughout this section, we set $\mathcal{V} = \mathrm{Alt}^{(2)}$ and $\mathfrak{N} = \mathrm{NAlt}^{(2)}$.

2.1 Free $\mathcal{V}$-algebras

Recall that by the Artin Theorem [28, Chap. 2.3] every two-generated alternative algebra is an associative one. It is also well known that every alternative algebra satisfies the central Moufang identity

$$x \cdot yz \cdot x = xy \cdot zx.$$

Thus by virtue of metability (4) in the free algebra $F_{\mathcal{V}}[X]$, we have

$$x \cdot yz \cdot x = 0.$$

Moreover, combining alternativity (1) with (4), we get

$$x(x \cdot yz) = 0, \quad (yz \cdot x)x = 0, \quad (x, zt, y) = (zt \cdot y)x = x(y \cdot zt).$$

Therefore the relations

$$T_xT_x^* = T_xT_x = 0, \quad [L_x, R_y] = R_yR_x = L_yL_x$$

hold for the operators of multiplication acting on $(F_{\mathcal{V}}[X])^2$. Using these relations, one can prove the following

Lemma 2.1 ([7, 15])**.** *The free algebra* $F_{\mathcal{V}}[X]$ *is a linear span of the monomials of the form*

$$(x_{i_1}x_{i_2})\,T_{x_{i_3}}R_{x_{i_4}}\ldots R_{x_{i_n}},$$

which are skew-symmetric with respect to their variables $x_{i_3}, x_{i_4}, \ldots, x_{i_n}$.

Note that in view of non-nilpotency of $F_{\mathcal{V}}[X]$ (see [5, 19]), Lemma 2.1 implies immediately the infiniteness of the basic rank of $\mathcal{V}$.

Further, by metability of $F_{\mathcal{V}}[X]$, the Artin Theorem yields the following

Proposition 2.1. *The algebra* $F_{\mathcal{V}}[X]$ *satisfies the identities*

$$x^3T_y = 0, \tag{8}$$

$$(xy)T_xR_y = 0. \tag{9}$$

Combining Lemmas 2.1 with identities (8), (9), it is not hard to prove the following

Lemma 2.2. $(F_{\mathcal{V}}[X_n])^{n+2} = 0$.

2.2 Free superalgebras on odd generators

Let us set

$$\varphi(x_1, x_2, x_3) = \sum_{\sigma\in \mathrm{S}_3} \left(x_{\sigma(1)}x_{\sigma(2)}\right) x_{\sigma(3)}.$$

Lemma 2.3. *The free metabelian superalgebra on one odd generator satisfies the superidentity* $\tilde{\varphi}(x_1, x_2, x_3) = 0$.

Proof. Let $\mathcal{A}$ be the free metabelian superalgebra on one odd generator. Consider the value $\tilde{\varphi}(x_1, x_2, x_3)$ for arbitrary homogeneous elements $x_1, x_2, x_3 \in \mathcal{A}$. By metability of $\mathcal{A}$, we may assume that at least two of the elements x_1, x_2, x_3 are generators of $\mathcal{A}$. But in this case, taking into account that $\mathcal{A}$ has only one odd generator, we obtain that the linear combination $\tilde{\varphi}(x_1, x_2, x_3)$ contains with every its monomial αw $(\alpha = \pm 1)$ the monomial $-\alpha w$. Hence, $\tilde{\varphi}(x_1, x_2, x_3) = 0$ in $\mathcal{A}$. □

Lemma 2.4. *The intersection* $\mathcal{I} = (x^3)^{\mathrm{T}} \cap \left(\sum_{n=3}^{\infty} \mathcal{P}_n(\mathcal{V})\right)$ *is spanned over* F *by the element* $\varphi(x_1, x_2, x_3)$.

Proof. It is clear that $\varphi(x_1, x_2, x_3) \in \mathcal{I}$ as a linearization of x^3. Furthermore, by (8), the element x^3 and all its linearizations lie in the annihilator of $F_{\mathcal{V}}[X]$. On the other hand, Lemma 2.1 and identity (4) yield $\varphi(w, x_2, x_3) = 0$ for every element $w \in F_{\mathcal{V}}[X]^2$. Therefore every element of $\mathcal{I}$ is proportional to $\varphi(x_1, x_2, x_3)$. □

Lemma 2.5. *The free* $\mathcal{V}$*-superalgebra on two odd generators is an* $\mathfrak{N}$*-superalgebra.*

Proof. Let $\mathcal{A}$ be the free $\mathcal{V}$-superalgebra on two odd generators. By Lemma 2.4, it suffices to check $\tilde{\varphi}(x_1, x_2, x_3) = 0$ assuming that x_1, x_2, x_3 are generators of $\mathcal{A}$. But in this case, at least two of the elements x_1, x_2, x_3 coincide. Hence, with the similar arguments as in Lemma 2.3, one can prove that $\tilde{\varphi}(x_1, x_2, x_3) = 0$. □

2.3 Auxiliary $\mathfrak{N}$-superalgebra on one odd generator

Let $U = F \cdot x$ be a superalgebra generated by an odd element x such that $x^2 = 0$. Consider a $\mathbb{Z}_2$-graded space $M = M_0 + M_1$ over F such that

$$M_i = F[\varepsilon] \cdot a_i, \quad i = 0, 1,$$

where $F[\varepsilon]$ is an algebraic extension of F with a primitive 3-th root of 1, i. e. such an element ε that $\varepsilon^2+\varepsilon+1 = 0$. It is clear that if $\varepsilon \in F$, then M is a 2-dimensional space over F, otherwise, a 4-dimensional one. We define on M a structure of an U-superbimodule such that the action of the element x is given by the equalities

$$a_i \cdot x = a_{1-i}, \quad x \cdot a_i = (i + \varepsilon)a_{1-i}, \quad i = 0, 1.$$

Consider the superalgebra $\mathcal{A} = U \dotplus M$ with the $\mathbb{Z}_2$-grading

$$\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1, \quad \mathcal{A}_0 = M_0, \quad \mathcal{A}_1 = U + M_1$$

and the multiplication

$$(u_1 + m_1)\,(u_2 + m_2) = u_1 m_2 + m_1 u_2, \quad u_1, u_2 \in U, \quad m_1, m_2 \in M.$$

This superalgebra is called the *split null extension of U by M* (see [23, 28]). It is known [19, 23] that $\mathcal{A}$ is an alternative superalgebra.

Lemma 2.6. *$\mathcal{A}$ is an $\mathfrak{N}$-superalgebra generated by one odd element.*

Proof. Let us show that $\mathcal{A}$ can be generated by the element $y = a_1 + x$. First we have

$$y^2 = (a_1 + x)^2 = a_1 \cdot x + x \cdot a_1 = (2 + \varepsilon)a_0.$$

Hence for $\varepsilon \in F$, we get

$$a_0 = \alpha y^2, \quad a_1 = \alpha y^2 \cdot y, \quad \text{where} \quad \alpha = \frac{1-\varepsilon}{3}.$$

Otherwise, we calculate

$$\begin{aligned} y^2 \cdot y &= (2 + \varepsilon)a_0 \cdot x = (2 + \varepsilon)a_1, \\ y \cdot y^2 &= x \cdot (2 + \varepsilon)a_0 = (2\varepsilon + \varepsilon^2)a_1 = (\varepsilon - 1)a_1. \end{aligned}$$

Considering the difference of the obtained equalities, we have

$$3a_1 = y^2 \cdot y - y \cdot y^2.$$

Therefore to express any element of M with y over F it suffices to use the relations

$$a_0 = a_1 \cdot y, \qquad \varepsilon a_1 = y \cdot a_0, \qquad \varepsilon a_0 = \varepsilon a_1 \cdot y.$$

To conclude the proof note that $\mathcal{A}$ is metabelian by definition. Hence by Lemma 2.3, it satisfies identity (5). $\square$

2.4 Additive basis of $\mathcal{P}_n(\mathfrak{N})$

Definition 2.1. The *basis words of* $\mathcal{P}_n(\mathfrak{N})$ are the polynomials of the following types:

$$\begin{aligned}&1)\ \ (x_1x_2)\,TR^{n-3},\\ &2)\ \ (x_1\circ x_i)\,TR^{n-3},\quad i=2,\dots,n.\end{aligned}$$

Lemma 2.7. *The space* $\mathcal{P}_n(\mathfrak{N})$ *is spanned by its basis words.*

Proof. Let u be an arbitrary monomial of $\mathcal{P}_n(\mathfrak{N})$. By Lemma 2.1 we may assume that u has the form

$$u=(x_ix_j)\,TR^{n-3},\quad i,j\in\{1,\dots,n\}\,.$$

Consider the linear span I_n of the polynomials of $\mathcal{P}_n(\mathfrak{N})$ of the form

$$(x_i\circ x_j)\,TR^{n-3}.$$

Using the linearization

$$\sum_{\sigma\in\mathrm{C}_3}\big(x_{\sigma(1)}\circ x_{\sigma(2)}\big)\,x_{\sigma(3)}=0$$

of identity (5) it is not hard to show that I_n is the linear span of basis words of type 2). On the other hand, the linearizations

$$x\,(T_y\circ T_z)=(y\circ z)\,T^*_x$$

of identities (1) imply that the monomial u is skew-symmetric with respect to all its variables modulo I_n. ☐

Lemma 2.8. *Any nontrivial linear combination of basis words of* $\mathcal{P}_n(\mathfrak{N})$ *is not an identity of* $\mathrm{G}(\mathcal{A})$.

Proof. Consider an arbitrary linear combination $f_n=g_n+h_n$ of basis words of $\mathcal{P}_n(\mathfrak{N})$, where

$$\begin{aligned}g_n&=\alpha\,(x_1x_2)\,R^{n-2}+\alpha'\,(x_1x_2)\,LR^{n-3},\\ h_n&=\sum_{i=2}^{n}\big(\beta_i\,(x_1\circ x_i)\,R^{n-2}+\beta_i'\,(x_1\circ x_i)\,LR^{n-3}\big),\end{aligned}$$

for some scalars $\alpha,\alpha',\beta_j,\beta_j'\in F$. Suppose that $\mathcal{A}$ satisfies the superidentity $\tilde{f}_n=0$. Then let us show that all the coefficients in f_n are zero.

Fix $i>2$ and make in the identity $\tilde{f}_n=0$ the substitution $x_i=a_0,\ x_j=x$ for all $j\neq i$. Then we will get

$$\beta_i(1+\varepsilon)a_0R^{n-3}-\beta_i'(1+\varepsilon)^2a_0R^{n-3}=0,$$

which gives $\beta_i=(1+\varepsilon)\beta_i'$ for all $i>2$. If we substitute $x_i=a_1$ instead of a_0, we will similarly get $\beta_i=-\varepsilon\beta_i'$, which implies that $\beta_i=\beta_i'=0$ for all $i>2$. Therefore, h_n has a form

$$h_n=\beta(x_1\circ x_2)R^{n-2}+\beta'(x_1\circ x_2)LR^{n-3},\ \ \beta,\beta'\in F.$$

Substituting now $x_i = a_0$, $i = 1, 2$ and $x_j = x$ for all $j \neq i$ we will get the two equalities

$$\alpha a_0 R^{n-3} - \alpha'(1+\varepsilon) a_0 R^{n-3} + H_0 = 0,$$
$$\varepsilon(\alpha a_0 R^{n-3} - \alpha'(1+\varepsilon) a_0 R^{n-3}) + H_0 = 0,$$

where $H_0 = \tilde{h}_n(a_0, x, \ldots, x) = \tilde{h}_n(x, a_0, \ldots, x)$. This implies that $\alpha = (1+\varepsilon)\alpha'$ and $H_0 = 0$. Similarly, making the substitutions $x_i = a_1$, $i = 1, 2$ and $x_j = x$ for all $j \neq i$ we get $\alpha = -\varepsilon\alpha'$ and $H_1 = 0$, where $H_1 = \tilde{h}_n(a_1, x, \ldots, x) = -\tilde{h}_n(x, a_1, \ldots, x)$. Thus, $\alpha = \alpha' = 0$. Finally, we have

$$H_0 = \beta(1+\varepsilon) a_0 R^{n-3} - \beta'(1+\varepsilon)^2 a_0 R^{n-3} = 0,$$
$$H_1 = -\beta\varepsilon a_1 R^{n-3} - \beta'\varepsilon^2 a_1 R^{n-3} = 0,$$

which implies that $\beta = \beta' = 0$. Therefore all the scalars in f_n are zero. □

By Lemmas 2.6–2.8, we obtain $\mathfrak{N}_{0,1} = \tilde{\mathfrak{N}}$.

2.5 Index of nilpotency of $F_{\mathfrak{N}}[X_n]$

Lemma 2.9. *The index of nilpotency of $F_{\mathfrak{N}}[X_n]$ is equal to $n+2$.*

Proof. By Lemma 2.2, the index of nilpotency of $F_{\mathfrak{N}}[X_n]$ is not more than $n+2$. Let us provide a nonzero element of $F_{\mathfrak{N}}[X_n]$ of degree $n+1$. Consider the monomial

$$w = x_1^2 R_{x_2} \ldots R_{x_n}.$$

Note that the linearization of w can be written as a basis word of type 2) of $\mathcal{P}_{n+1}(\mathfrak{N})$. Hence by Lemmas 2.6 and 2.8, $w \neq 0$ in $F_{\mathfrak{N}}[X_n]$. □

Lemma 2.9 implies the strict inclusions $\mathfrak{N}_n \subset \mathfrak{N}_{n+1}$ for $n \in \mathbb{N}$.
Theorem 1 is proved.

2.6 Auxiliary $\mathcal{V}$-superalgebras

Let $\mathcal{A}$ be the superalgebra defined in Section 2.3. Consider two superalgebras $\mathcal{B} = \mathcal{B}_0 + \mathcal{B}_1$ and $\mathcal{B}' = \mathcal{B}'_0 + \mathcal{B}'_1$ defined by the following conditions:

1. $$\begin{aligned} \mathcal{B}_0 &= \mathcal{A}_0 + F\cdot e, & \mathcal{B}_1 &= \mathcal{A}_1 + F\cdot ex + F\cdot xe + F\cdot exe, \\ \mathcal{B}'_0 &= \mathcal{A}_0 + F\cdot yx + F\cdot xz, & \mathcal{B}'_1 &= \mathcal{A}_1 + F\cdot y + F\cdot z + F\cdot yxz; \end{aligned}$$

2. $\mathcal{A}$ is a subalgebra of $\mathcal{B}$ and $\mathcal{B}'$;

3. all nonzero products of basis elements of $\mathcal{B}$ and $\mathcal{B}'$, in the case when at least one of the factors doesn't lie in $\mathcal{A}$, are the following:

$$e\cdot x = ex, \quad x\cdot e = xe, \quad ex\cdot e = e\cdot xe = exe;$$
$$y\cdot x = yx, \quad x\cdot z = xz, \quad yx\cdot z = y\cdot xz = yxz.$$

Lemma 2.10. *$\mathcal{B}$ is a $\mathcal{V}$-superalgebra generated by one even and one odd elements; $\mathcal{B}'$ is a $\mathcal{V}$-superalgebra generated by three odd elements.*

Proof. It is clear that $\mathcal{B}$ can be generated by the elements e, x and $\mathcal{B}'$ can be generated by the elements x, y, z. Thus by virtue of Lemma 2.6, it remains to notice that if some product $\rho \neq 0$ of three basis elements of $\mathcal{B}$ or $\mathcal{B}'$ contains a factor not lying in $\mathcal{A}$, then ρ is associative and lies in the annihilator of the corresponding algebra $\mathcal{B}$ or $\mathcal{B}'$. $\square$

2.7 Additive basis of $\mathcal{P}_n(\mathcal{V})$

Definition 2.2. The *basis words of* $\mathcal{P}_n(\mathcal{V})$ are all basis words of $\mathcal{P}_n(\mathfrak{N})$ and the polynomial $\varphi(x_1, x_2, x_3)$.

It follows easily from Lemmas 2.4 and 2.7 that the space $\mathcal{P}_n(\mathcal{V})$ is spanned by its basis words.

Lemma 2.11. *Any nontrivial linear combination of basis words of $\mathcal{P}_n(\mathcal{V})$ is not an identity of either $\mathrm{G}(\mathcal{B})$ or $\mathrm{G}(\mathcal{B}')$.*

Proof. By definition of $\mathrm{G}(\mathcal{B})$ and $\mathrm{G}(\mathcal{B}')$, taking into account Lemma 2.8, it suffices to check that $\tilde{\varphi}(x_1, x_2, x_3)$ takes a nonzero value on some elements of $\mathcal{B}$ and $\mathcal{B}'$. Indeed,

$$\tilde{\varphi}(e, x, e) = 2\,(e \cdot x) \cdot e = 2\,exe \neq 0; \qquad \tilde{\varphi}(y, x, z) = (y \cdot x) \cdot z = yxz \neq 0.$$

$\square$

By Lemmas 2.10 and 2.11, we have $\mathcal{V}_{1,1} = \mathcal{V}_{0,3} = \tilde{\mathcal{V}}$. Combining Lemmas 2.5 and 2.11 with Theorem 1, we obtain $\mathcal{V}_{0,2} \subseteq \tilde{\mathfrak{N}} = \mathcal{V}_{0,1} \subset \tilde{\mathcal{V}}$.

2.8 Strictness of inclusions $\mathcal{V}_n \subset \mathcal{V}_{n+1}$

Finally, to complete the proof of Theorem 2, it suffices to verify the strictness of all inclusions in the first row of the lattice $\mathcal{L}(\mathcal{V})$. In fact, it follows from Lemmas 2.4 and 2.9 that the index of nilpotency of $F_{\mathcal{V}}[X_n]$ for $n \geqslant 2$ is equal to $n+2$. Therefore, $\mathcal{V}_n \neq \mathcal{V}_{n+1}$ for all $n \geqslant 2$. It remains to notice that the variety $\mathcal{V}_1$ is commutative and $\mathcal{V}_2$ is not.

Theorem 2 is proved.

3 Jordan algebras

Throughout this section, we set $\mathcal{V} = \mathrm{Jord}^{(2)}$.

3.1 Additive basis of the free $\mathcal{V}$-algebra

It is known [4, 9, 15, 22] that an additive basis of the free algebra $F_{\mathcal{V}}\,[X]$ can be formed by the following monomials

$$(x_k x_{i_1})\, R_{x_{j_1}} R_{x_{i_2}} R_{x_{j_2}} \dots R_{x_{i_t}} R'_{x_{j_t}},$$

where

$$k \geqslant i_1 < i_2 < \cdots < i_t, \quad j_1 < j_2 < \cdots < j_t,$$

and the symbol $'$ means that the operator $R_{x_{j_t}}$ is absent when the degree of monomial is even. In what follows, we use this basis with no comments.

3.2 Nilpotency of the free algebra $F_{\mathcal{V}}\,[X_n]$

Proposition 3.1. *The algebra $F_{\mathcal{V}}\,[X]$ satisfies the identities*

$$x^2 R_y R_x = 0, \tag{10}$$

$$(zt) R_x R_y R_x = 0, \tag{11}$$

$$(zx) R_y R_x R_t R_z = 0. \tag{12}$$

Proof. First we stress that (10) is a direct consequence of metability and (2). Further, taking into account commutativity, we write the partial linearization of (10) in the form

$$2(zx) R_y R_x + x^2 R_y R_z = 0. \tag{13}$$

Then by setting $z := zt$ in (13), in view of metability, we get (11). Finally, multiplying (13) by $R_t R_z$ and applying (11), we obtain (12). $\square$

Lemma 3.1. *The algebra $F_{\mathcal{V}}\,[X_n]$ is nilpotent of index $2n+2$.*

Proof. Identities (10)–(12) imply immediately that $(F_{\mathcal{V}}\,[X_n])^{2n+2} = 0$. Therefore it remains to note that the element

$$x_1^2 R_{x_1} R_{x_2}^2 R_{x_3}^2 \dots R_{x_n}^2$$

of the additive basis of $F_{\mathcal{V}}\,[X]$ is a nonzero element of $(F_{\mathcal{V}}\,[X_n])^{2n+1}$. $\square$

Lemma 3.1 yields that all inclusions $\mathcal{V}_n \subset \mathcal{V}_{n+1}$ are strict and, consequently, the basic rank of $\mathcal{V}$ is infinite.

3.3 Estimate of basic superrank of $\mathcal{V}$

First note that superizing (11), one can prove the following

Proposition 3.2. *Let $F_{\mathcal{V}}^{(s)}\,[Z]$ be a free $\mathcal{V}$-superalgebra on an arbitrary set Z of even and odd generators. Then the operators of multiplication acting on $\left(F_{\mathcal{V}}^{(s)}\,[Z]\right)^2$ satisfy the relation*

$$R_x R_y R_z = (-1)^{|x||y|+|x||z|+|y||z|+1} R_z R_y R_x, \tag{14}$$

where $x, y, z \in Z$ and $|x|$ denotes the parity of x.

Let $U = U_0 + U_1$ be the superalgebra

$$U_0 = \{0\}, \quad U_1 = F \cdot x + F \cdot y$$

with null multiplication and $M = M_0 + M_1$ be the vector space

$$M_0 = F \cdot a, \quad M_1 = F \cdot v.$$

Consider a split null extension $\mathcal{A} = U \dotplus M$ with a supercommutative multiplication such that all nonzero products of the basis elements of $\mathcal{A}$, up to the order of factors, are the following:

$$a \cdot x = v, \quad v \cdot y = a.$$

The supercommutativity rule means that even elements commute with all elements of superalgebra but products of two odd elements are anticommutative.

Lemma 3.2. *$\mathcal{A}$ is a $\mathcal{V}$-superalgebra generated by two odd elements.*

Proof. By definition, $\mathcal{A}$ is metabelian, supercommutative, and can be generated by the elements $v + x$ and y. It remains to check that $\mathcal{A}$ is a Jordan superalgebra. Note that by construction of $\mathcal{A}$ it suffices to verify that relation (14) holds in $\mathcal{A}$. But it follows trivially from the definition of multiplication in $\mathcal{A}$.[1] $\square$

Lemma 3.3. *The variety $\mathcal{V}$ is generated by* $\mathrm{G}(\mathcal{A})$.

Proof. In view of Lemma 3.2 it suffices to prove that $\mathrm{G}(\mathcal{A})$ doesn't satisfy any nontrivial identity in $\mathcal{V}$. Consider an arbitrary linear combination of basis monomials of $\mathcal{P}_n(\mathcal{V})$:

$$f_n = \sum_I \alpha_I (x_k x_{i_1}) R_{x_{j_1}} R_{x_{i_2}} R_{x_{j_2}} \dots R_{x_{i_t}} R'_{x_{j_t}},$$

where $t = \left[\frac{n}{2}\right]$ and I runs all possible sets $k, i_1, \dots, i_t, j_1, \dots, j_t$ of indices such that

$$k > i_1 < i_2 < \dots < i_t, \quad j_1 < j_2 < \dots < j_t.$$

Suppose that $\mathcal{A}$ satisfies the superidentity $\tilde{f}_n = 0$. Let us show that all the scalars in f_n are zero. We fix $I = \{k, i_1, \dots, i_t, j_1, \dots, j_t\}$ and make the substitution

$$x_k = a, \quad x_{i_1} = \dots = x_{i_t} = x, \quad x_{j_1} = \dots = x_{j_t} = y.$$

Then it is not hard to see that $\tilde{f}_n$ turns out to be proportional with the coefficient $\pm\alpha_I$ to the element

$$(a \cdot x) R_y R_x \dots = \begin{cases} v, & \text{if } n \text{ is even,} \\ a, & \text{if } n \text{ is odd} \end{cases}$$

that is nonzero in $\mathcal{A}$. Therefore, $\alpha_I = 0$. $\square$

Lemma 3.3 implies that $\mathcal{V}_{0,2} = \tilde{\mathcal{V}}$. Thus, to complete the proof of Theorem 3, it remains to show that the chain

$$\mathcal{V}_{0,1} \subset \mathcal{V}_{1,1} \subset \mathcal{V}_{2,1} \subset \dots \subset \mathcal{V}_{r,1} \subset \dots \subset \tilde{\mathcal{V}}$$

ascends strictly and all the inclusions $\mathcal{V}_{n,0} \subset \mathcal{V}_{n,1}$ are also strict.

[1] One can also note that $\mathcal{A}$ is a special Jordan superalgebra isomorphic to a subalgebra of the matrix superalgebra $M_{2,2}^{(+)}$ (see [19]) with the generators $e_{13} - e_{24} + e_{31} - e_{42}$ and $e_{13} + e_{14} + e_{24} + e_{31} + e_{32} + e_{42}$.

3.4 Strictness of the inclusions $\mathcal{V}_{n,0} \subset \mathcal{V}_{n,1}$ and $\mathcal{V}_{n-1,1} \subset \mathcal{V}_{n,1}$

Let $U^{(n)} = U_0^{(n)} + U_1^{(n)}$ be the superalgebra

$$U_0^{(n)} = \sum_{i=1}^{n} F \cdot e_i, \quad U_1^{(n)} = F \cdot y$$

with null multiplication and $\mathcal{A}^{(n)} = \mathcal{A}_0^{(n)} + \mathcal{A}_1^{(n)}$ be an associative superalgebra with the unit $\mathbf{1}$ generated by the even elements $\mathbf{1}, \mathbf{e}_1, \ldots, \mathbf{e}_n$ and one odd element $\mathbf{y}$ with the defining relations

$$\mathbf{e}_i\mathbf{e}_j = 0, \quad \mathbf{y}^2 = \mathbf{1}, \quad \mathbf{e}_i\mathbf{y}\mathbf{e}_j = -\mathbf{e}_j\mathbf{y}\mathbf{e}_i.$$

Consider a split null extension $\mathcal{B}^{(n)} = U^{(n)} \dot{+} \mathcal{A}^{(n)}$ with a supercommutative multiplication induced by the actions

$$\mathbf{1} \cdot e_i = 0, \quad \mathbf{a} \cdot e_i = \mathbf{a}\mathbf{e}_i, \quad \mathbf{1} \neq \mathbf{a} \in \mathcal{A}^{(n)},$$
$$\mathbf{b} \cdot y = \mathbf{b}\mathbf{y}, \quad \mathbf{b} \in \mathcal{A}^{(n)}.$$

Lemma 3.4. *$\mathcal{B}^{(n)}$ is a $\mathcal{V}$-superalgebra generated by n even and one odd elements.*

Proof. By definition, $\mathcal{B}^{(n)}$ is metabelian and supercommutative. Moreover, it is not hard to see that $\mathcal{B}^{(n)}$ can be generated by the elements $\mathbf{1} + e_1, e_2, \ldots, e_n, y$. Thus it remains to prove that $\mathcal{B}^{(n)}$ is a Jordan superalgebra. Actually by definition of multiplication in $\mathcal{B}^{(n)}$, taking into account that $\left(\mathcal{B}^{(n)}\right)^2 \subseteq \mathcal{A}^{(n)}$, it suffices to verify that relation (14) holds for the operators of the form $R_{u_1}R_{u_2}R_{u_3}$ acting on $\mathcal{A}^{(n)}$ for $u_i \in \{e_1, \ldots, e_n, y\}$. Indeed, besides the trivial case $u_1 = u_3 = y$, we check that $R_{e_i}R_{e_j}$ annihilates $\mathcal{A}^{(n)}$:

$$\mathbf{a}R_{e_i}R_{e_j} = \mathbf{a}\mathbf{e}_i\mathbf{e}_j = 0;$$

R_y^2 acts on $\mathcal{A}^{(n)}$ identically:

$$\mathbf{a}R_y^2 = \mathbf{a}\mathbf{y}^2 = \mathbf{a};$$

and the action of $R_{e_i}R_yR_{e_j}$ on $\mathcal{A}^{(n)}$ is skew-symmetric with respect to e_i, e_j:

$$\mathbf{a}R_{e_i}R_yR_{e_j} = \mathbf{a}\mathbf{e}_i\mathbf{y}\mathbf{e}_j = -\mathbf{a}\mathbf{e}_j\mathbf{y}\mathbf{e}_i = -\mathbf{a}R_{e_j}R_yR_{e_i}.$$

□

By Lemmas 3.1 and 3.4, in view of non-nilpotency of $\mathcal{B}^{(n)}$, we obtain $\mathcal{V}_{n,0} \subset \mathcal{V}_{n,1}$.

Further, let us denote by $f_n = f_n(a, b, x_1, \ldots, x_{2n})$ the polynomial

$$f_n = (ab)\left(R_{x_1} \circ R_{x_2}\right) \ldots \left(R_{x_{2n-1}} \circ R_{x_{2n}}\right),$$

where $R_x \circ R_y = R_xR_y + R_yR_x$.

Lemma 3.5. *The free $\mathcal{V}$-superalgebra on $n-1$ even and one odd generators satisfies the superidentity $\tilde{f}_n = 0$.*

Proof. Let $A_{n-1,1}$ be the free $\mathcal{V}$-superalgebra on $n-1$ even and one odd generators. Consider a value of $\tilde{f}_n$ on some homogeneous elements $\tilde{a}, \tilde{b}, \tilde{x}_1, \ldots, \tilde{x}_{2n} \in A_{n-1,1}$. In view of metability, we may assume that all the elements of the set $S = \{\tilde{x}_1, \ldots, \tilde{x}_{2n}\}$ are generators of $A_{n-1,1}$. Thus by definition of f_n, it is clear that $\tilde{f}_1 = 0$ in $A_{0,1}$.

Let us prove by induction on n that $\tilde{f}_n = 0$ in $A_{n-1,1}$. For $n \geqslant 2$, by $e_1, \ldots, e_{n-1}$ we denote the even generators of $A_{n-1,1}$ and y denotes its odd generator. For a monomial $w = (\tilde{a}\tilde{b})R_{\tilde{x}_1}R_{\tilde{x}_2}\ldots R_{\tilde{x}_{2n}}$ we set $S(w) = \{\tilde{x}_1, \tilde{x}_3, \ldots, \tilde{x}_{2n-1}\}$ and $\bar{S}(w) = S \backslash S(w)$. Assume that $w \neq 0$ in $A_{n-1,1}$. Then it follows from (14) that every e_i can be presented only once in each set $S(w)$ and $\bar{S}(w)$. Thus, $\tilde{f}_n$ can be nonzero only if every e_i is included in S not more than twice. On the other hand, if some e_i is included in S only once, then one can represent $\tilde{f}_n$ in the form

$$\tilde{f}_n = \pm \tilde{f}_{n-1}\big(\tilde{a}, \tilde{b}, \tilde{x}_1, \ldots, \tilde{x}_{2j-2}, \tilde{x}_{2j+1}, \ldots, \tilde{x}_{2n-2}\big)\left(R_{\tilde{x}_{2j-1}} \circ R_{\tilde{x}_{2j}}\right),$$

where $e_i \in \{\tilde{x}_{2j-1}, \tilde{x}_{2j}\}$. In this case, by inductive hypothesis, we have $\tilde{f}_n = 0$. Therefore, it suffices to consider the case when every e_i is included in S twice exactly and $w \neq 0$. This assumption yields that the sets $S(w)$ and $\bar{S}(w)$ consist of the same elements $e_1, \ldots, e_{n-1}, y$. Consequently, except of w, there is only one more nonzero monomial w' in the linear combination $\tilde{f}_n$ of the form

$$w' = (\tilde{a}\tilde{b})R_{\tilde{x}_2}R_{\tilde{x}_1}R_{\tilde{x}_4}R_{\tilde{x}_3}\ldots R_{\tilde{x}_{2n}}R_{\tilde{x}_{2n-1}}.$$

Hence we have

$$\tilde{f}_n = w + (-1)^{|\tilde{x}_1||\tilde{x}_2| + |\tilde{x}_3||\tilde{x}_4| + \cdots + |\tilde{x}_{2n-1}||\tilde{x}_{2n}|}w' = w + w'.$$

By virtue of (14), taking into account the equalities $S(w') = \bar{S}(w)$ and $\bar{S}(w') = S(w)$, it is not hard to see that w' is proportional to w. Thus it remains to prove that the coefficient of this proportionality is equal to -1. Let σ be the permutation that transforms $\bar{S}(w)$ into $S(w)$. It is clear that one can transform w' into w acting by σ on $S(w')$ and by σ^{-1} on $\bar{S}(w')$. While that, a scalar ± 1 appearing after this transformation will not depend on the parity of σ, but will depend only on number of transpositions made by the odd elements. Taking into account that $\sigma(y) \neq y$, it is not hard to understand that such a transposition will be only one. Therefore, $w' = -w$ and, consequently, $\tilde{f}_n = 0$. $\square$

By Lemmas 3.4 and 3.5, to prove the strictness of inclusions $\mathcal{V}_{n-1,1} \subset \mathcal{V}_{n,1}$ it suffices to verify that $\tilde{f}_n$ takes a nonzero value in $\mathcal{B}^{(n)}$. Indeed,

$$\begin{aligned}\tilde{f}_n\left(\mathbf{1}, y, e_1, y, e_2, y, \ldots, e_n, y\right) &= \mathbf{y}\left(R_{e_1} \circ R_y\right)\left(R_{e_2} \circ R_y\right)\ldots\left(R_{e_n} \circ R_y\right) = \\ &= \left(\mathbf{y}\mathbf{e}_1 \cdot y + \mathbf{1} \cdot e_1\right)\left(R_{e_2} \circ R_y\right)\ldots\left(R_{e_n} \circ R_y\right) = \\ &= \mathbf{y}\mathbf{e}_1\mathbf{y}\left(R_{e_2} \circ R_y\right)\ldots\left(R_{e_n} \circ R_y\right) = \cdots = \mathbf{y}\mathbf{e}_1\mathbf{y}\mathbf{e}_2\mathbf{y}\ldots\mathbf{e}_n\mathbf{y} \neq 0.\end{aligned}$$

Theorem 3 is proved.

Remark 3.1. Note that Theorem 3 gives a more detailed description of the inclusions in the lattice $\mathcal{L}(\mathcal{V})$ than one needs to deduce the uniqueness of the basic superrank $(0,2)$ for $\mathcal{V}$. Actually proving that $\mathrm{sp_b}(\mathcal{V}) = \{(0,2)\}$ we could restrict with establishing the equality $\mathcal{V}_{0,2} = \tilde{\mathcal{V}}$ and the strict inclusions $\mathcal{V}_{r,1} \subset \tilde{\mathcal{V}}$ for $r = 0, 1, 2, \ldots$ Namely, in view of Lemmas 3.2 and 3.5, it is enough to check that the superpolynomial $\tilde{f}_n$ doesn't vanish on some elements of the superalgebra $\mathcal{A}$.

4 Malcev algebras

Throughout this section, we set $\mathcal{V} = \mathrm{Malc}^{(2)}$.

4.1 Preliminary identities

It's well-known [17] that an anticommutative algebra over a field of characteristic distinct from 2 is a Malcev one if and only if it satisfies the Sagle identity

$$\sum_{\sigma\in\mathrm{C}_4} \left(x_{\sigma(1)}x_{\sigma(2)}\right) R_{x_{\sigma(3)}} R_{x_{\sigma(4)}} = \left(x_1x_3\right)\left(x_2x_4\right).$$

By virtue of metability, the Sagle identity gets the form

$$\sum_{\sigma\in\mathrm{C}_4} \left(x_{\sigma(1)}x_{\sigma(2)}\right) R_{x_{\sigma(3)}} R_{x_{\sigma(4)}} = 0. \tag{15}$$

For $x_4 = w \in \left(F_{\mathcal{V}}\left[X\right]\right)^2$, identity (15) implies

$$wR_{x_1}R_{x_2}R_{x_3} = wR_{x_3}R_{x_1}R_{x_2}. \tag{16}$$

Moreover, combining (15) with anticommutativity and taking into account that $\operatorname{char} F \neq 2$, we obtain

$$(xy)\left[R_x, R_y\right] = 0. \tag{17}$$

Finally, applying (16) and (17), we have

$$(xy)\rho R_x\eta R_y = (xy)\rho R_y\eta R_x, \tag{18}$$

for any operator words ρ, η.

4.2 Auxiliary $\mathcal{V}$-superalgebra

Let $U = U_0 + U_1$ be the superalgebra

$$U_0 = F\cdot e, \quad U_1 = F\cdot y$$

with null multiplication and $M = M_0 + M_1$ be the vector space

$$M_0 = F\cdot a, \quad M_1 = F\cdot v + F\cdot w.$$

Consider a split null extension $\mathcal{A} = U \dotplus M$ with a superanticommutative multiplication such that all nonzero products of the basis elements of $\mathcal{A}$, up to the order of factors, are the following:

$$a\cdot y = v, \quad v\cdot y = a, \quad w\cdot e = w.$$

The superanticommutativity rule means that even elements anticommute with all elements of superalgebra and odd elements commute to each other.

Lemma 4.1. *$\mathcal{A}$ is a $\mathcal{V}$-superalgebra on one even and one odd generators.*

Proof. By definition, $\mathcal{A}$ is metabelian and superanticommutative. Moreover, it is not hard to see that $\mathcal{A}$ can be generated by the even element $a+e$ and by the odd element $w+y$. It remains to check that $\mathcal{A}$ is a Malcev superalgebra. By construction of $\mathcal{A}$, it suffices to verify that the superization of (16) holds in $\mathcal{A}$, i. e. that the action of an operator $R_{z_1}R_{z_2}R_{z_3}$ on M, for homogeneous $z_i \in U$, satisfies the relation

$$R_{z_1}R_{z_2}R_{z_3} = (-1)^{|z_1||z_2|+|z_1||z_3|}R_{z_2}R_{z_3}R_{z_1}.$$

Actually it is not hard to see that this relation holds trivially in both possible nonzero cases $z_1 = z_2 = z_3 = y$ and $z_1 = z_2 = z_3 = e$. □

4.3 Additive basis of $\mathcal{P}_n(\mathcal{V})$

By virtue of anticommutativity, we consider the space $\mathcal{P}_n(\mathcal{V})$ only for $n \geqslant 3$. Following the fixed above notations, we write down the monomials of $\mathcal{P}_n(\mathcal{V})$ omitting some uniquely restored indices that are assumed to be arranged in the ascending order.

Definition 4.1. The *basis words of* $\mathcal{P}_n(\mathcal{V})$ are the polynomials of the following types:

1) $(x_1x_2)R^{n-2}, \quad (x_2x_3)R^{n-2}, \quad (x_3x_1)R^{n-2}$,
2) $(x_1x_i)R^{n-2}, \quad i = 4, \ldots, n$,
3) $(x_1x_2)(R_{x_3} \circ R_{x_4})R^{n-4}, \quad (x_2x_3)(R_{x_1} \circ R_{x_4})R^{n-4}, \quad (x_3x_1)(R_{x_2} \circ R_{x_4})R^{n-4}$,
4) $(x_1x_i)(R_{x_2} \circ R_{x_3})R^{n-4}, \quad i = 4, \ldots, n$,

where $R_x \circ R_y = R_xR_y + R_yR_x$.

Lemma 4.2. *$\mathcal{P}_n(\mathcal{V})$ is spanned by its basis words.*

Proof. In view of anticommutativity, it is clear that $\mathcal{P}_3(\mathcal{V})$ is spanned by its basis words of type 1).

Let I_n be the linear span of basis words of $\mathcal{P}_n(\mathcal{V})$ for $n \geqslant 4$. Then using (15) and anticommutativity, we have

$$\begin{aligned}(x_2x_4)R_{x_1}R_{x_3} = -(x_4x_1)R_{x_3}R_{x_2} - (x_1x_3)R_{x_2}R_{x_4} - (x_3x_2)R_{x_4}R_{x_1} = \\ (x_1x_4)(R_{x_2} \circ R_{x_3}) - (x_1x_4)R_{x_2}R_{x_3} - (x_1x_3)R_{x_2}R_{x_4} + \\ + (x_2x_3)(R_{x_1} \circ R_{x_4}) - (x_2x_3)R_{x_1}R_{x_4} \equiv 0 \pmod{I_4}.\end{aligned}$$

Similarly, it is not hard to check that

$$(x_2x_4)R_{x_3}R_{x_1},\ (x_3x_4)R_{x_1}R_{x_2},\ (x_3x_4)R_{x_2}R_{x_1} \in I_4.$$

Thus, $I_4 = \mathcal{P}_4(\mathcal{V})$.

Further, let u be an arbitrary monomial of $\mathcal{P}_n(\mathcal{V})$ for $n \geqslant 5$. Then applying (16) one can order the indices of variables of u as follows:

$$u = (x_{i_1}x_{i_2})R_{x_{i_3}} \ldots R_{x_{i_n}}, \quad i_3 < i_5, \quad i_4 < \cdots < i_n.$$

Thus, similarly to the case $n = 4$, combining (15) with anticommutativity and (16), one can obtain $u \in I_n$. □

Lemma 4.3. *Any nontrivial linear combination of basis words of $\mathcal{P}_n(\mathcal{V})$ is not an identity of* $\mathrm{G}(\mathcal{A})$.

Proof. Consider first the case $n = 3$. We set

$$f = \alpha\,(x_1x_2)\,x_3 + \beta\,(x_2x_3)\,x_1 + \gamma\,(x_3x_1)\,x_2, \quad \alpha, \beta, \gamma \in F$$

and assume that $\tilde{f} = 0$ in $\mathcal{A}$. Then by the substitution $x_1 = a,\ x_2 = x_3 = y$, we have

$$\tilde{f} = \alpha(a \cdot y) \cdot y - \gamma(y \cdot a) \cdot y = (\alpha + \gamma)a = 0.$$

Similarly, after two more substitutions $x_2 = a,\ x_1 = x_3 = y$ and $x_3 = a,\ x_1 = x_2 = y$, we get the system

$$\begin{cases} \alpha + \gamma = 0, \\ \alpha + \beta = 0, \\ \beta + \gamma = 0. \end{cases}$$

The unique solution of this system in F is $\alpha = \beta = \gamma = 0$.

Now consider a linear combination

$$f_n = g_n + h_n + p_n + q_n$$

of basis words of $\mathcal{P}_n(\mathcal{V})$ for $n \geqslant 4$, where

$$g_n = \alpha_1\,(x_1x_2)\,R^{n-2} + \alpha_2\,(x_2x_3)\,R^{n-2} + \alpha_3\,(x_3x_1)\,R^{n-2},$$

$$h_n = \sum_{i=4}^{n} \alpha_i\,(x_1x_i)\,R^{n-2},$$

$$p_n = \beta_1\,(x_1x_2)\,(R_{x_3} \circ R_{x_4})\,R^{n-4} + \beta_2\,(x_2x_3)\,(R_{x_1} \circ R_{x_4})\,R^{n-4} + \beta_3\,(x_3x_1)\,(R_{x_2} \circ R_{x_4})\,R^{n-4},$$

$$q_n = \sum_{i=4}^{n} \beta_i\,(x_1x_i)\,(R_{x_2} \circ R_{x_3})\,R^{n-4}, \quad \alpha_i, \beta_i \in F.$$

Suppose that $\mathcal{A}$ satisfies the superidentity $\tilde{f}_n = 0$. Then let us show that all the coefficients in f_n are zero.

First we fix $i \geqslant 4$ and make in $\tilde{f}_n = 0$ the substitution $x_i = a,\ x_j = y$ for all $j \neq i$. Then we get

$$\alpha_i(y \cdot a)R_y^{n-2} = \begin{cases} -\alpha_i v, & \text{if } n \text{ is even}, \\ -\alpha_i a, & \text{if } n \text{ is odd}, \end{cases}$$

whence, $\alpha_i = 0$ for $i \geqslant 4$ and

$$f_n = g_n + p_n + q_n.$$

Similarly, for $i \geqslant 4$, by the substitution $x_i = w,\ x_j = e$ for all $j \neq i$, one can prove that $\beta_i = 0$. Thus,

$$f_n = g_n + p_n.$$

Further, for $i = 1, 2, 3$, by the substitution $x_i = a,\ x_j = y$ for all $j \neq i$, similarly to the case $n = 3$, we obtain $\alpha_i = 0$. Consequently,

$$f_n = p_n.$$

Finally, for $i = 1, 2, 3$, by the substitution $x_i = w,\ x_j = e$ for all $j \neq i$, we get $\beta_i = 0$. $\square$

By Lemmas 4.1–4.3, we have $\mathcal{V}_{1,1} = \tilde{\mathcal{V}}$. Thus, to complete the proof of Theorem 4, it remains to show that the chains

$$\mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}_r \subset \cdots \subset \mathcal{V}, \qquad \mathcal{V}_{0,1} \subset \mathcal{V}_{0,2} \subset \cdots \subset \mathcal{V}_{0,s} \subset \cdots \subset \tilde{\mathcal{V}}$$

ascend strictly.

4.4 Strictness of inclusions $\mathcal{V}_n \subset \mathcal{V}_{n+1}$

Let G_n be the Grassmann algebra with the unit $\mathbf{1}$ on the set $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of anticommuting generators and U_n be the algebra on the set $e_1, \ldots, e_n$ of generators with null multiplication. Consider a split null extension $A_n = U_{n-1} \dotplus \mathrm{G}_{n-1}$ with an anticommutative multiplication induced by the actions

$$(\mathbf{e}_i \ldots \mathbf{e}_j) \cdot e_k = \mathbf{e}_i \ldots \mathbf{e}_j \mathbf{e}_k.$$

Lemma 4.4. *A_n is a $\mathcal{V}_n$-algebra.*

Proof. By definition, A_n is metabelian and anticommutative. Moreover, it is not hard to see that A_n can be generated by the elements $\mathbf{1}, e_1, \ldots, e_{n-1}$. Thus it remains to check that A_n satisfies (15). Indeed, for arbitrary $w \in \mathrm{G}_n$, we have

$$(w \cdot e_i) R_{e_j} R_{e_k} + (e_k \cdot w) R_{e_i} R_{e_j} = w \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k - w \mathbf{e}_k \mathbf{e}_i \mathbf{e}_j = 0.$$

□

Further, let us denote by $f_n = f_n(x_1, \ldots, x_n)$ the polynomial

$$f_n = \sum_{\sigma \in \mathrm{S}_n} (-1)^{|\sigma|} \left(x_{\sigma(1)} x_{\sigma(2)}\right) R_{x_{\sigma(3)}} \ldots R_{x_{\sigma(n)}}.$$

Lemma 4.5. *The free algebra $F_{\mathcal{V}}[X_n]$ satisfies the identity $f_{n+1} = 0$.*

Proof. By definition, the value of f_n is zero when values of two of its variables coincide. Hence, by virtue of metability, it remains to verify that f_{n+1} takes zero value after a substitution $x_{n+1} = w$, where $w = (x_i x_j)\tau$ for some operator word τ. Indeed, in view of anticommutativity and metability, we have

$$f_{n+1}(x_1, \ldots, x_n, w) = (-1)^n 2 w \chi_n, \qquad \chi_n = \sum_{\sigma \in \mathrm{S}_n} (-1)^{|\sigma|} R_{x_{\sigma(1)}} \ldots R_{x_{\sigma(n)}}.$$

Thus, on one hand, the action of χ_n is skew-symmetric with respect to any pair of its variables by definition. On the other hand, by virtue of (18), χ_n acts at w symmetrically w.r.t. x_i, x_j. Therefore, this action is zero. □

In view of Lemmas 4.4 and 4.5, to prove the strict inclusions $\mathcal{V}_n \subset \mathcal{V}_{n+1}$ it suffices to check that f_{n+1} takes a nonzero value in A_{n+1}. Indeed,

$$f_{n+1}(\mathbf{1}, e_1, \ldots, e_n) = 2 \sum_{\sigma \in \mathrm{S}_n} (-1)^{|\sigma|} \mathbf{1}\, R_{e_{\sigma(1)}} \ldots R_{e_{\sigma(n)}} = 2n!\, \mathbf{e}_1 \ldots \mathbf{e}_n \neq 0.$$

Remark 4.1. V. T. Filippov [6] proved the strict inclusion $\mathrm{Malc}_n \subset \mathrm{Malc}_{n+1}$ for all $n \neq 3$ and suggested the hypothesis $\mathrm{Malc}_3 = \mathrm{Malc}_4$. This hypothesis is still known as a difficult open problem.

4.5 Strictness of inclusions $\mathcal{V}_{0,n} \subset \mathcal{V}_{0,n+1}$

Proposition 4.1. *The free $\mathcal{V}$-superalgebra $F_{\mathcal{V}}^{(s)}[Y]$ on an arbitrary set Y of odd generators satisfies the relations*

$$wR_xR_yR_z = wR_zR_xR_y, \tag{19}$$

$$(xy)\rho R_x\eta R_y = (xy)\rho R_y\eta R_x, \tag{20}$$

$$x^2\rho R_x\eta R_y = x^2\rho R_y\eta R_x, \tag{21}$$

where $x, y, z \in Y$, $w \in \left(F_{\mathcal{V}}^{(s)}[Y]\right)^2$, and ρ, η are arbitrary operator words of the form $R_{y_i} \dots R_{y_j}$, $y_i, \dots, y_j \in Y$.

Proof. Superizing (15) and taking into account metability and superanticommutativity, we have

$$\begin{aligned} wR_xR_yR_z + (-1)^{|w|}wL_zR_xR_y &= wR_xR_yR_z - wR_zR_xR_y = 0, \\ (xy)R_xR_y - (yx)R_yR_x + (xy)R_xR_y - (yx)R_yR_x &= 2(xy)\left[R_x, R_y\right] = 0, \\ x^2R_xR_y - x^2R_yR_x + (xy)R_x^2 - (yx)R_x^2 &= x^2\left[R_x, R_y\right] = 0. \end{aligned}$$

Thus, (19) is proved and to conclude the proof of (20), (21) it remains to combine (19) with the last two obtained relations. □

Further, let us denote by $g_n = g_n(a, b, x_1, \dots, x_n)$ the polynomial

$$g_n = (ab)\sum_{\sigma \in \mathrm{S}_n} R_{x_{\sigma(1)}} \dots R_{x_{\sigma(n)}}.$$

Lemma 4.6. *The free $\mathcal{V}$-superalgebra $F_{\mathcal{V}}^{(s)}[Y_n]$ on a finite set $Y_n = \{y_1, \dots, y_n\}$ of odd generators satisfies the superidentity $\tilde{g}_n = 0$.*

Proof. By definition, the value of $\tilde{g}_n$ is zero when values of at least two of its variables $x_1, \dots, x_n$ coincide. Hence, by virtue of metability, it suffices to consider the case when $\tilde{g}_n$ takes a value

$$\tilde{g}_n = (ab)\chi_n, \qquad \chi_n = \sum_{\sigma \in \mathrm{S}_n} (-1)^{|\sigma|} R_{y_{\sigma(1)}} \dots R_{y_{\sigma(n)}}$$

for some $a, b \in F_{\mathcal{V}}^{(s)}[Y_n]$. Thus, on one hand, the action of χ_n is skew-symmetric with respect to any pair of its variables by definition. On the other hand, by virtue of (20) and (21), χ_n acts at ab symmetrically with respect to some pair of its variables. Therefore, this action is zero. □

To conclude the proof of strict inclusions $\mathcal{V}_{0,n} \subset \mathcal{V}_{0,n+1}$ let us construct a $\mathcal{V}$-superalgebra on a set of $n+1$ odd generators that does not satisfy the superidentity $\tilde{g}_n = 0$.

Let $\mathrm{G}^{(n)}$ be the Grassmann algebra with the unit $\mathbf{1}$ on the set $e_1, \dots, e_n$ of anticommuting generators and $\bar{\mathrm{G}}^{(n)}$ be the Grassmann algebra without unit on the set $\bar{e}_1, \dots, \bar{e}_n$ of anticommuting generators. For an element $w \in \mathrm{G}^{(n)}$ of the form $w = e_{i_1} \dots e_{i_k} \neq \mathbf{1}$

we use the notations $|w|=k$, $\bar{w}=\bar{e}_{i_1}\dots\bar{e}_{i_k}$, and set $|\mathbf{1}|=0$, $\bar{\mathbf{1}}=0$. For $i=0,1$, by $\mathrm{G}_i^{(n)}$ and $\bar{\mathrm{G}}_i^{(n)}$ we denote, respectively, the subspaces of $\mathrm{G}^{(n)}$ and $\bar{\mathrm{G}}^{(n)}$ spanned by all the words w and $\bar{w}$ such that $|w|\equiv i \pmod 2$. Consider the direct some $M^{(n)}=\mathrm{G}^{(n)}+\bar{\mathrm{G}}^{(n)}$ of vector spaces, where we set $M_0^{(n)}=\mathrm{G}_0^{(n)}+\bar{\mathrm{G}}_1^{(n)}$ to be an even component of $M^{(n)}$ and $M_1^{(n)}=\mathrm{G}_1^{(n)}+\bar{\mathrm{G}}_0^{(n)}$ to be its odd component.

Let $U^{(n)}$ be the superalgebra on the set $y_1,\dots,y_n$ of odd generators with null multiplication. Consider a split null extension $\mathcal{A}^{(n)}=U^{(n)}\dotplus M^{(n)}$ with a superanticommutative multiplication induced by the actions

$$w\cdot y_k=we_k,\quad \bar{w}\cdot y_k=\bar{w}\bar{e}_k,\quad w\in\mathrm{G}^{(n)}.$$

By virtue of skew-symmetry of the elements of $\mathrm{G}^{(n)}$ and $\bar{\mathrm{G}}^{(n)}$ with respect to their generators it is not hard to prove the following

Lemma 4.7. *$\mathcal{A}^{(n)}$ is a $\mathcal{V}$-superalgebra.*

Further, we consider an extension $\bar{\mathcal{A}}^{(n+1)}=F\cdot x+\mathcal{A}^{(n)}$ of the superalgebra $\mathcal{A}^{(n)}$ such that

$$\bar{\mathcal{A}}_0^{(n+1)}=\mathcal{A}_0^{(n)},\qquad \bar{\mathcal{A}}_1^{(n+1)}=F\cdot x+\mathcal{A}_1^{(n)},$$

and all nonzero products of basis elements with x are the following:

$$x^2=\mathbf{1},\qquad x\cdot y_i=y_i\cdot x=\bar{e}_i,\qquad x\cdot\bar{w}=(-1)^{|w|}\bar{w}\cdot x=\frac{1}{2}w,\quad \bar{w}\in\bar{\mathrm{G}}^{(n)}.$$

Lemma 4.8. *$\bar{\mathcal{A}}^{(n+1)}$ is a $\mathcal{V}$-superalgebra generated by $n+1$ odd element.*

Proof. By Lemma 4.7, taking into account skew-symmetry of the elements of $\mathrm{G}^{(n)}$ and $\bar{\mathrm{G}}^{(n)}$ with respect to their generators, it suffices to verify that the superization of (15) holds in the following cases:

$$x^2R_{y_i}R_{y_j}-(x\cdot y_i)R_{y_j}R_x-(y_j\cdot x)R_xR_{y_i}=(\mathbf{1}\cdot y_i)\cdot y_j-(\bar{e}_i\cdot y_j)\cdot x-(\bar{e}_j\cdot x)\cdot y_i=$$
$$=e_i\cdot y_j-\bar{e}_i\bar{e}_j\cdot x+\frac{1}{2}e_j\cdot y_i=e_ie_j-\frac{1}{2}e_ie_j+\frac{1}{2}e_je_i=0,$$

$$(x\cdot y_i)R_xR_{y_j}-(y_i\cdot x)R_{y_j}R_x+(x\cdot y_j)R_xR_{y_i}-(y_j\cdot x)R_{y_i}R_x=$$
$$=(\bar{e}_i\cdot x)\cdot y_j-(\bar{e}_i\cdot y_j)\cdot x+(\bar{e}_j\cdot x)\cdot y_i-(\bar{e}_j\cdot y_i)\cdot x=$$
$$=-\frac{1}{2}e_i\cdot y_j-\bar{e}_i\bar{e}_j\cdot x-\frac{1}{2}e_j\cdot y_i-\bar{e}_j\bar{e}_i\cdot x=-\frac{1}{2}e_ie_j-\frac{1}{2}e_ie_j-\frac{1}{2}e_je_i-\frac{1}{2}e_je_i=0,$$

$$(x\cdot\bar{w})R_{y_i}R_{y_j}-(-1)^{|w|}(\bar{w}\cdot y_i)R_{y_j}R_x=$$
$$=\frac{1}{2}(w\cdot y_i)\cdot y_j-(-1)^{|w|}\bar{w}\bar{e}_i\bar{e}_j\cdot x=\frac{1}{2}we_ie_j-\frac{1}{2}we_ie_j=0,$$

$$(\bar{w}\cdot x)R_{y_i}R_{y_j} - (-1)^{|w|}(y_j\cdot\bar{w})R_xR_{y_i} = \frac{(-1)^{|w|}}{2}(w\cdot y_i)\cdot y_j - (\bar{w}\bar{e}_j\cdot x)\cdot y_i =$$
$$= \frac{(-1)^{|w|}}{2}we_ie_j - \frac{(-1)^{|w|+1}}{2}we_j\cdot y_i = \frac{(-1)^{|w|}}{2}\left(we_ie_j + we_je_i\right) = 0,$$

$$(\bar{w}\cdot y_i)R_xR_{y_j} - (-1)^{|w|}(y_j\cdot\bar{w})R_{y_i}R_x = (\bar{w}\bar{e}_i\cdot x)\cdot y_j - (\bar{w}\bar{e}_j\cdot y_i)\cdot x =$$
$$= \frac{(-1)^{|w|+1}}{2}we_i\cdot y_j - \bar{w}\bar{e}_j\bar{e}_i\cdot x = -\frac{(-1)^{|w|}}{2}\left(we_ie_j + we_je_i\right) = 0.$$

$\square$

In view of Lemmas 4.6–4.8, to prove the strict inclusions $\mathcal{V}_{0,n}\subset\mathcal{V}_{0,n+1}$ it remains to check that $\tilde{g}_n$ takes a nonzero value in $\bar{\mathcal{A}}^{(n+1)}$. Indeed,

$$\tilde{g}_n\left(x,x,y_1,\ldots,y_n\right) = \sum_{\sigma\in\mathrm{S}_n}(-1)^{|\sigma|}\mathbf{1}\,R_{y_{\sigma(1)}}\ldots R_{y_{\sigma(n)}} = n!\,e_1\ldots e_n\neq 0.$$

Theorem 4 is proved.

Remark 4.2. We stress that the description of the inclusions in the lattice $\mathcal{L}(\mathcal{V})$ obtained in Theorem 4 is more detailed than one needs to deduce the uniqueness of the basic superrank $(1,1)$ for $\mathcal{V}$. Actually proving that $\mathrm{sp_b}(\mathcal{V}) = \{(1,1)\}$ we could restrict with establishing the equality $\mathcal{V}_{1,1}=\tilde{\mathcal{V}}$ and the strict inclusions $\mathcal{V}_n\subset\mathcal{V}$, $\mathcal{V}_{0,n}\subset\tilde{\mathcal{V}}$. Namely, in view of Lemmas 4.1, 4.5, and 4.6, it is enough to check that the superpolynomials $\tilde{f}_n$ $\tilde{g}_n$ don't vanish on some elements of the superalgebra $\mathcal{A}$.

Remark 4.3. It is not hard to check that in fact the superalgebra $\bar{\mathcal{A}}^{(n+1)}$ is a Lie superalgebra. The variety $\mathrm{Lie}^{(2)}$ of metabelian Lie algebras has a finite basic rank: it is generated by a 2–dimensional non-abelian algebra and hence $r_b\big(\mathrm{Lie}^{(2)}\big)=2$. Since we are interested in metabelian varieties of infinite basic rank, it was not planed to consider the variety $\mathrm{Lie}^{(2)}$. But in view of the results of this section we notice that the lattices $\mathcal{L}\big(\mathrm{Lie}^{(2)}\big)$ and $\mathcal{L}\big(\mathrm{Malc}^{(2)}\big)$ have a quite similar structure in spite of the difference in their initial chains. Indeed, one can easily prove that $\mathrm{Lie}^{(2)}$ also possesses the basic superrank $(1,1)$. Moreover, the strict inclusions $\mathrm{Lie}^{(2)}_{0,n}\subset\mathrm{Lie}^{(2)}_{0,n+1}$ are provided by the fact that the free metabelian Lie superalgebra on n odd generators is nilpotent of index $n+2$. Thus the lattice $\mathcal{L}(\mathcal{V})$, for $\mathcal{V}=\mathrm{Lie}^{(2)}$, has the form

$$\begin{array}{ccccccccc}
 & & \mathcal{V}_{1,0} & \subset & \mathcal{V}_{2,0} & =\cdots= & \mathcal{V}_{r,0} & =\cdots \\
 & & \cap & & \shortparallel & & \shortparallel & \\
\mathcal{V}_{0,1} & \subset & \mathcal{V}_{1,1} & = & \mathcal{V}_{2,1} & =\cdots= & \mathcal{V}_{r,1} & =\cdots \\
\cap & & \shortparallel & & \shortparallel & & \shortparallel & \\
\mathcal{V}_{0,2} & \subset & \mathcal{V}_{1,2} & = & \mathcal{V}_{2,2} & =\cdots= & \mathcal{V}_{r,2} & =\cdots \\
\cap & & \shortparallel & & \shortparallel & & \shortparallel & \\
\mathcal{V}_{0,3} & \subset & \mathcal{V}_{1,3} & = & \mathcal{V}_{2,3} & =\cdots= & \mathcal{V}_{r,3} & =\cdots \\
\cap & & \shortparallel & & \shortparallel & & \shortparallel & \\
\multicolumn{8}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}
\end{array}$$

In particular, $\mathrm{sp_b}\big(\mathrm{Lie}^{(2)}\big)=\big\{(2,0),(1,1)\big\}$.

5 Metabelian algebras

5.1 Endomorphisms corresponding to Young tables

Let d be a *Young diagram* of order n and τ be a permutation of S_n. Following [1, Chap. 3], by τd we denote the *Young table* obtained by filing in the diagram d with the numbers $\tau(1), \ldots, \tau(n)$ in the order from the top down and from left to right. The group S_n acts naturally on the set of all tables corresponding to d: $\sigma(\tau d) = (\sigma\tau)d$. By $\mathrm{C}_{\tau d}$ we denote the *column stabilizer of the table* τd, i. e. the subgroup of S_n consisting of all permutations preserving the set of symbols in each column of τd. Similarly, one can define the *row stabilizer* $\mathrm{R}_{\tau d}$ *of* τd.

By $\mathcal{P}_n$ we denote the *linear span of all associative multilinear words of degree* n *on the set* X_n *of variables.* For every Young table τd of order $n' \leqslant n$ we define the following endomorphisms:

$$\varphi_{\tau d}, \psi_{\tau d} : \mathcal{P}_n \mapsto \mathcal{P}_n,$$
$$\varphi_{\tau d}\,(w) = \sum_{\sigma \in \mathrm{C}_{\tau d}} \sum_{\rho \in \mathrm{R}_{\sigma\tau d}} (-1)^{|\sigma|} w\left(x_{\rho\sigma(1)}, \ldots, x_{\rho\sigma(n')}, x_{n'+1}, \ldots, x_n\right),$$
$$\psi_{\tau d}\,(w) = \sum_{\rho \in \mathrm{R}_{\tau d}} \sum_{\sigma \in \mathrm{C}_{\rho\tau d}} (-1)^{|\sigma|} w\left(x_{\sigma\rho(1)}, \ldots, x_{\sigma\rho(n')}, x_{n'+1}, \ldots, x_n\right).$$

For example, if $\tau d = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$ and $w = x_1x_2x_3x_4$, then $\varphi_{\tau d}\,(w)$ can be calculated as follows:

$$\varphi_{\tau d}(w) = \sum_{\rho \in \mathrm{R}_{\tau d}} x_{\rho(1)}x_{\rho(2)}x_{\rho(3)}x_{\rho(4)} - \sum_{\rho \in \mathrm{R}_{(13)\tau d}} x_{\rho(3)}x_{\rho(2)}x_{\rho(1)}x_{\rho(4)} -$$
$$- \sum_{\rho \in \mathrm{R}_{(24)\tau d}} x_{\rho(1)}x_{\rho(4)}x_{\rho(3)}x_{\rho(2)} + \sum_{\rho \in \mathrm{R}_{(13)(24)\tau d}} x_{\rho(3)}x_{\rho(4)}x_{\rho(1)}x_{\rho(2)}$$

and, taking into account that

$$(13)\tau d = \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & 4 \\ \hline\end{array}, \qquad (24)\tau d = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 & 2 \\ \hline\end{array}, \qquad (13)(24)\tau d = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array},$$

we get

$$\varphi_{\tau d}(w) = (x_1 \circ x_2)(x_3 \circ x_4) - (x_3 \circ x_2)(x_1 \circ x_4) -$$
$$- (x_1 \circ x_4)(x_3 \circ x_2) + (x_3 \circ x_4)(x_1 \circ x_2) =$$
$$= (x_1 \circ x_2) \circ (x_3 \circ x_4) - (x_3 \circ x_2) \circ (x_1 \circ x_4).$$

Similarly, for the same table τd and $w' = x_1x_3x_2x_4$, we have

$$\psi_{\tau d}\,(w') = \sum_{\sigma \in \mathrm{C}_{\tau d}} (-1)^{|\sigma|} x_{\sigma(1)}x_{\sigma(3)}x_{\sigma(2)}x_{\sigma(4)} + \sum_{\sigma \in \mathrm{C}_{(12)\tau d}} (-1)^{|\sigma|} x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(1)}x_{\sigma(4)} +$$
$$+ \sum_{\sigma \in \mathrm{C}_{(34)\tau d}} (-1)^{|\sigma|} x_{\sigma(1)}x_{\sigma(4)}x_{\sigma(2)}x_{\sigma(3)} + \sum_{\sigma \in \mathrm{C}_{(12)(34)\tau d}} (-1)^{|\sigma|} x_{\sigma(2)}x_{\sigma(4)}x_{\sigma(1)}x_{\sigma(3)}$$

and, observing that

$$(12)\tau d = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & 4 \\ \hline\end{array}\,, \qquad (34)\tau d = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline\end{array}\,, \qquad (12)(34)\tau d = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 4 & 3 \\ \hline\end{array}\,,$$

we obtain

$$\psi_{\tau d}(w') = [x_1, x_3][x_2, x_4] + [x_2, x_3][x_1, x_4] + [x_1, x_4][x_2, x_3] + [x_2, x_4][x_1, x_3] =$$
$$= [x_1, x_3] \circ [x_2, x_4] + [x_2, x_3] \circ [x_1, x_4].$$

We stress that, by definition, the polynomial $\varphi_{\tau d}(w)$ is skew-symmetric with respect to each pair of its variables whose indices stand at the same column of τd. Similarly, $\psi_{\tau d}(w)$ is symmetric with respect to each pair of its variables whose indices stand at the same row of τd.

In what follows, the introduced endomorphisms are considered for rectangular Young tables only. The notation $k \times m$ means that a table has k rows and m columns.

By $\mathfrak{A}_{r,s}$ denote the *free associative superalgebra on the set* $Y_{r,s}$ *of* r *even and* s *odd generators.*

Lemma 5.1. *Let* $\xi : \mathcal{P}_n \mapsto \mathfrak{A}_{r,s}$ *be a homomorphism induced by a mapping* $\xi : X_n \mapsto Y_{r,s}$, *where* $n \geqslant (r+1)(rs+s+1)$. *Then for every rectangular* $(r+1) \times (rs+s+1)$ *table* τd *and every word* $w \in \mathcal{P}_n$, *the superization of the polynomial* $\xi(\varphi_{\tau d}(w))$ *is equal to zero.*

Proof. By definition of the polynomial $f = \varphi_{\tau d}(w)$, we have $f = \sum_{\sigma \in \mathrm{C}_{\tau d}} (-1)^{|\sigma|} f'_\sigma$, where

$$f'_\sigma = \sum_{\rho \in \mathrm{R}_{\sigma\tau d}} w\left(x_{\rho\sigma(1)}, \ldots, x_{\rho\sigma(n')}, x_{n'+1}, \ldots, x_n\right), \quad n' = (r+1)(rs+s+1).$$

Let us substitute the element i by $\xi(x_i)$ in each cell of the table τd. The obtained table we denote by $(\tau d)^\xi$.

First, in view of skew-symmetry of f w.r.t. each pair of its variables whose indices stand at the same column of τd, we note that $\widetilde{\xi(f)}$ can be non-zero only if none of the columns of $(\tau d)^\xi$ contains the same even generator of $\mathfrak{A}_{r,s}$ twice. Consequently, the whole number of even elements in $(\tau d)^\xi$ is not more then $r(rs+s+1)$.

Further, if $\widetilde{\xi(f)} \neq 0$, then we may assume that $\widetilde{\xi(f'_\sigma)} \neq 0$ at least for one $\sigma \in \mathrm{C}_{\tau d}$. In this case, we stress that f'_σ is symmetric w.r.t. each pair of its variables whose indices stand at the same row of $\sigma\tau d$. Hence, none of the rows of $(\sigma\tau d)^\xi$ contains the same odd element twice. Consequently, the whole number of odd elements in $(\tau d)^\xi$ is not more then $(r+1)s$.

Therefore, the number of elements in $(\tau d)^\xi$ is not more then

$$r(rs+s+1) + (r+1)s = (r+1)(rs+s+1) - 1.$$

The obtained contradiction completes the proof. □

By the similar arguments one can prove the following

Lemma 5.2. *Let* $\xi : \mathcal{P}_n \mapsto \mathfrak{A}_{r,s}$ *be a homomorphism induced by a mapping* $\xi : X_n \mapsto Y_{r,s}$, *where* $n \geqslant (rs+r+1)(s+1)$. *Then for every rectangular* $(rs+r+1) \times (s+1)$ *table* τd *and every word* $w \in \mathcal{P}_n$, *the superization of* $\xi(\psi_{\tau d}(w))$ *is equal to zero.*

5.2 Inclusions in the lattice $\mathcal{L}(\mathfrak{M})$

Lemma 5.3. *For all naturals r, s, we have $\mathfrak{M}_r \not\subseteq \mathfrak{M}_{r-1,s}$.*

Proof. Let $f_k = f_k(u, v, x_1, \ldots, x_{kr})$ be the polynomial

$$f_k = (uv) \sum_{\sigma \in \mathrm{C}_{\tau d}} \sum_{\rho \in \mathrm{R}_{\sigma\tau d}} (-1)^{|\sigma|} R_{x_{\rho\sigma(1)}} \ldots R_{x_{\rho\sigma(kr)}},$$

where

$$\tau d = \begin{array}{|c|c|c|c|} \hline 1 & r+1 & \ldots & (k-1)r+1 \\ \hline 2 & r+2 & \ldots & (k-1)r+2 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline r & 2r & \ldots & kr \\ \hline \end{array} .$$

We stress that applying Lemma 5.1, it is not hard to prove that $\tilde{f}_k = 0$ is a superidentity in $\mathfrak{M}_{r-1,s}$ for $k = rs + 1$. Thus to prove the Lemma it suffices to construct some $\mathfrak{M}_r$-algebra $A^{(r)}$ such that f_k takes nonzero values in $A^{(r)}$ for all k.

Consider the algebra

$$U^{(r)} = \sum_{i=1}^{r} F \cdot e_i$$

with null multiplication and the vector space

$$M^{(r)} = \sum_{n \in \mathbb{Z}_r} F \cdot a_n.$$

Let $A^{(r)} = U^{(r)} \dotplus M^{(r)}$ be the split null extension with the multiplication induced by the following actions:

$$a_n \cdot e_i = \begin{cases} a_{n+1 \pmod r}, & n \equiv i \pmod r, \\ 0, & n \not\equiv i \pmod r. \end{cases}$$

By definition, $A^{(r)}$ is a metabelian algebra generated by the elements $a_{\bar{0}} + e_r, e_1, \ldots, e_{r-1}$.

Consider the mapping $\xi : X_{kr} \mapsto \{e_1, \ldots, e_r\}$ defined by the $r \times k$ table

$$(\tau d)^{\xi} = \begin{array}{|c|c|c|c|} \hline e_1 & e_1 & \ldots & e_1 \\ \hline e_2 & e_2 & \ldots & e_2 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline e_r & e_r & \ldots & e_r \\ \hline \end{array},$$

i. e. $\xi(x_i)$ is the element of the $(\tau d)^{\xi}$ standing in the cell corresponding to the index i in τd. To conclude the proof it remains to verify that f_k takes a nonzero value in $A^{(r)}$. Indeed,

$$f_k(a_{\bar{0}}, e_r, \xi(x_1), \ldots, \xi(x_{kr})) = k!\, r\, a_{\bar{1}} \neq 0.$$

□

Lemma 5.4. *For all naturals r, s, we have $\mathfrak{M}_{0,s} \not\subseteq \mathfrak{M}_{r,s-1}$.*

Proof. Let $f_k = f_k(u, v, x_1, \dots, x_{ks})$ be the polynomial

$$f_k = (uv) \sum_{\rho \in \mathrm{R}_{\tau d}} \sum_{\sigma \in \mathrm{C}_{\rho\tau d}} (-1)^{|\sigma|} R_{x_{\sigma\rho(1)}} \dots R_{x_{\sigma\rho(ks)}},$$

where

$$\tau d = \begin{array}{|c|c|c|c|} \hline 1 & 2 & \dots & s \\ \hline s+1 & s+2 & \dots & 2s \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline (k-1)s+1 & (k-1)s+2 & \dots & ks \\ \hline \end{array}.$$

We stress that applying Lemma 5.2, it is not hard to prove that $\tilde{f}_k = 0$ is a superidentity in $\mathfrak{M}_{r,s-1}$ for $k = rs + 1$. Thus to prove the Lemma it suffices to construct some $\mathfrak{M}_{0,s}$-superalgebra $\mathcal{A}^{(s)}$ such that $\tilde{f}_k$ takes nonzero values in $\mathcal{A}^{(s)}$ for all k.

Let $U^{(s)} = U_0^{(s)} + U_1^{(s)}$ be the superalgebra

$$U_0^{(s)} = \{0\}, \qquad U_1^{(s)} = \sum_{i=1}^{s} F \cdot y_i$$

with null multiplication and $M^{(s)} = M_0^{(s)} + M_1^{(s)}$ be the vector space

$$M_0^{(s)} = \sum_{n \in \mathbb{Z}_s} F \cdot a_{2n}, \qquad M_1^{(s)} = \sum_{n \in \mathbb{Z}_s} F \cdot a_{2n+1}.$$

Consider a split null extension $\mathcal{A}^{(s)} = U^{(s)} \dotplus M^{(s)}$ with a multiplication induced by the following actions:

$$a_n \cdot y_i = \begin{cases} a_{n+1 \pmod{2s}}, & n \equiv i \pmod{s}, \\ 0, & n \not\equiv i \pmod{s}. \end{cases}$$

One can easily check that $\mathcal{A}^{(s)}$ is a metabelian superalgebra generated by the odd elements $a_{\bar{1}} + y_1, y_2, \dots, y_s$.

Let us consider the mapping $\xi : X_{ks} \mapsto \{y_1, \dots, y_s\}$ defined by the $k \times s$ table

$$(\tau d)^{\xi} = \begin{array}{|c|c|c|c|} \hline y_1 & y_2 & \dots & y_s \\ \hline y_1 & y_2 & \dots & y_s \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline y_1 & y_2 & \dots & y_s \\ \hline \end{array},$$

i. e. $\xi(x_i)$ is the element of the $(\tau d)^{\xi}$ standing in the cell corresponding to the index i in τd. To conclude the proof it remains to verify that $\tilde{f}_k$ takes a nonzero value in $\mathcal{A}^{(s)}$. Indeed,

$$\frac{1}{k!s} \tilde{f}_k(a_{\bar{0}}, y_s, \xi(x_1), \dots, \xi(x_{ks})) = \begin{cases} a_{\bar{1}}, & \text{if } k \text{ is even}, \\ a_{\overline{s+1}}, & \text{if } k \text{ is odd}. \end{cases}$$

□

Lemmas 5.3 and 5.4 yield that $\mathfrak{M}_{r',s} \not\subseteq \mathfrak{M}_{r,s'}$ and $\mathfrak{M}_{r,s'} \not\subseteq \mathfrak{M}_{r',s}$ for all nonnegative integers $r' < r$, $s' < s$. Moreover, the inclusions $\mathfrak{M}_{r',s} \subset \mathfrak{M}_{r,s}$ and $\mathfrak{M}_{r,s'} \subset \mathfrak{M}_{r,s}$ are strict.

Theorem 5 is proved.

Corollary 5.1 is an immediate consequence of Theorem 5.

5.3 Variety of unique arbitrary given finite basic superrank

Let us deduce Corollary 5.2. For an arbitrary pare $(r,s) \neq (0,0)$ of nonnegative integers, consider the free $\mathfrak{M}_{r,s}$-superalgebra $\mathfrak{A}_{r,s}$. Let $\mathfrak{N} = \mathrm{Var}\,\mathrm{G}\,(\mathfrak{A}_{r,s})$ be the variety generated by the Grassmann envelope of $\mathfrak{A}_{r,s}$. Then, by definition, we have $\tilde{\mathfrak{N}} = \mathfrak{M}_{r,s}$. It remains to prove that $\mathfrak{N}$ doesn't possess any other basic superrank (r',s') such that at least one of the inequalities $r' < r$, $s' < s$ holds. Indeed, we stress that Theorem 5 states the strict inclusion

$$(\mathfrak{M}_{r,s} \cap \mathfrak{M}_{r',s'}) \subset \mathfrak{M}_{r,s}.$$

At the same time, it is clear that $\mathfrak{N}_{r',s'} = \mathfrak{M}_{r,s} \cap \mathfrak{M}_{r',s'}$. Thus, $\mathfrak{N}_{r',s'} \subset \tilde{\mathfrak{N}}$, i. e. $(r',s') \notin \mathrm{sp_b}\,(\mathfrak{N})$. Therefore, $\mathrm{sp_b}\,(\mathfrak{N}) = \{(r,s)\}$.

6 Right alternative and right symmetric algebras

The variety of right alternative algebras is a well-known source of examples of nonfinitely based varieties [2, 8, 12, 16] over a field of characteristic zero. The similar results can be obtained for the variety of right symmetric algebras. In this section, we construct the varieties of right alternative and right symmetric algebras having no finite basic superrank.

Recall that by $\mathcal{V}^{\langle\varepsilon\rangle}$ $(\varepsilon = \pm 1)$ we denote the subvariety of $\mathfrak{M}$ distinguished by identities (6) and (7). Let us prove Theorem 6.

6.1 Auxiliary $\mathcal{V}^{\langle\varepsilon\rangle}$-superalgebra

Let $U = U_0 + U_1$ be the superalgebra

$$U_0 = \{0\}, \quad U_1 = \sum_{i=1}^{\infty} F \cdot y_i$$

with null multiplication and $M = M_0 + M_1$ be the vector space

$$M_0 = \sum_{i=1}^{\infty} F \cdot a_i, \quad M_1 = \sum_{i=1}^{\infty} F \cdot w_i.$$

Consider a split null extension $\mathcal{A}^{\langle\varepsilon\rangle} = U \dotplus M$ such that all nonzero products of the basis elements of $\mathcal{A}^{\langle\varepsilon\rangle}$ are the following:

$$y_i \cdot a_i = \varepsilon\, a_i \cdot y_i = w_{i+1}, \quad y_i \cdot w_i = a_i.$$

Lemma 6.1. *$\mathcal{A}^{\langle\varepsilon\rangle}$ is a $\mathcal{V}^{\langle\varepsilon\rangle}$-superalgebra on a countable set of odd generators.*

Proof. By definition, $\mathcal{A}^{\langle\varepsilon\rangle}$ is metabelian and can be generated by the elements $w_1, y_1, y_2, \ldots$. Moreover, it is not hard to check that the only nonzero associators on the basis elements of $\mathcal{A}^{\langle\varepsilon\rangle}$ are the following:

$$\begin{aligned}
(y_i, w_i, y_i) &= (y_i \cdot w_i) \cdot y_i = a_i \cdot y_i = \varepsilon w_{i+1}, \\
(y_i, y_i, w_i) &= -y_i \cdot (y_i \cdot w_i) = -y_i \cdot a_i = -w_{i+1}, \\
(y_{i+1}, a_i, y_i) &= -y_{i+1} \cdot (a_i \cdot y_i) = -y_{i+1} \cdot (\varepsilon w_{i+1}) = -\varepsilon a_{i+1}, \\
(y_{i+1}, y_i, a_i) &= -y_{i+1} \cdot (y_i \cdot a_i) = -y_{i+1} \cdot w_{i+1} = -a_{i+1}.
\end{aligned}$$

Thus one can see that the superization of (6) holds in $\mathcal{A}^{\langle\varepsilon\rangle}$. Finally, it remains to prove that $\mathcal{A}^{\langle\varepsilon\rangle}$ satisfies the superization of (7). Actually it suffices to notice that $\langle M_0, U\rangle_\varepsilon = 0$. □

6.2 Infiniteness of the basic superrank of $\mathcal{V}^{\langle\varepsilon\rangle}$

Let $f_{k,n} = f_{k,n}\left(u, v, x_1, z_1, x_2, z_2, \ldots, x_{kn-1}, z_{kn-1}, x_{kn}\right)$ be the polynomial

$$f_{k,n} = (uv) \sum_{\rho \in \mathrm{R}_{\tau d}} \sum_{\sigma \in \mathrm{C}_{\rho\tau d}} (-1)^{|\sigma|} L_{x_{\sigma\rho(1)}} L_{z_1} L_{x_{\sigma\rho(2)}} L_{z_2} \ldots L_{x_{\sigma\rho(kn-1)}} L_{z_{kn-1}} L_{x_{\sigma\rho(kn)}},$$

where

$$\tau d = \begin{array}{|c|c|c|c|}
\hline
1 & 2 & \ldots & n \\
\hline
n+1 & n+2 & \ldots & 2n \\
\hline
\vdots & \vdots & \ddots & \vdots \\
\hline
(k-1)n+1 & (k-1)n+2 & \ldots & kn \\
\hline
\end{array}.$$

By Lemma 5.2, the superpolynomial $\tilde{f}_{k,n}$ is a superidentity of $\mathfrak{M}_{r,s}$ for $k = rs + r + 1$ and $n = s + 1$. Therefore in view of Lemma 6.1, to prove the strictness of inclusions $\mathcal{V}^{\langle\varepsilon\rangle}_{r,s} \subset \tilde{\mathcal{V}}^{\langle\varepsilon\rangle}$ it suffices to verify that $\tilde{f}_{k,n}$ takes a nonzero value in $\mathcal{A}^{\langle\varepsilon\rangle}$. Indeed, let us denote by

$$\lambda_{k,n} = \lambda_{k,n}\left(L_{x_1}, L_{z_1}, L_{x_2}, L_{z_2}, \ldots, L_{x_{kn-1}}, L_{z_{kn-1}}, L_{x_{kn}}\right)$$

the superpolynomial on operators of left multiplication such that $\tilde{f}_{k,n} = (uv)\lambda_{k,n}$. Then by the substitution $u = y_1$, $v = w_1$, $x_i = y_i$, and $z_i = y_{i+1}$ for $i = 1, \ldots, kn$, we have

$$\tilde{f}_{k,n} = a_1 \lambda_{k,n}\left(L_{y_1}, L_{y_2}, L_{y_2}, L_{y_3}, \ldots, L_{y_{kn-1}}, L_{y_{kn}}, L_{y_{kn}}\right) = w_2 L_{y_2}^2 L_{y_3}^2 \ldots L_{y_{kn}}^2 = w_{kn+1} \neq 0.$$

Theorem 6 is proved.

7 Open problems

1. Is it true that for every pare of nonnegative integers $(r, s) \neq (0, 0)$ there is a variety $\mathcal{V}$ of associative algebras that has the unique basic superrank (r, s)?

2. What condition should satisfy a set of $n \geqslant 2$ pares of nonnegative integers to be the basic spectrum of some variety of algebras?

3. Do the varieties Alt, Jord, Malc have finite basic superranks?

4. Does every solvable subvariety of Alt, Jord, Malc have a finite basic superrank?

5. Are there any subvarieties of Alt, Jord, Malc of infinite basic superrank?

6. Does every Spechtian variety of algebras have a finite basic superrank?

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